J. Ritter, Augsburg
Iwasawa 2004, Besançon, July 5-9, 2004

Toward nonabelian equivariant Iwasawa theory, 2 ¶

I better begin by recalling some data from the previous talk. K_∞/k was a Galois extension of totally real number fields satisfying K_∞ *contains the cyclotomic* l*-extension* k_∞ *of* k *(where* l *is a given odd prime number) and* K_∞/k_∞ *is finite.* The corresponding Galois groups were $G_\infty = G_{K_\infty/k}$, $H = G_{K_\infty/k_\infty} \lhd G_\infty$, $G_\infty/H = \Gamma_k = G_{k_\infty/k}$. With the groups G_∞, Γ_k we had their completed group rings $\Lambda(G_\infty)$ and $\Lambda(\Gamma_k)$ over $\mathbb{Z}_l$ and the total rings of fractions of these, $\mathcal{Q}(G_\infty), \mathcal{Q}(\Gamma_k)$. Moreover, $\mathcal{Q}^{\mathrm{c}}(\Gamma_k)$ was short for ${\mathbb{Q}_l}^{\mathrm{c}} \otimes_{\mathbb{Q}_l} \mathcal{Q}(\Gamma_k)$, and $R_l(G_\infty)$ denoted the additive group generated by all ${\mathbb{Q}_l}^{\mathrm{c}}$-valued characters χ of G_∞ with open kernel. And there was this part of the localization sequence of K-theory and the Det map

$$\begin{array}{ccccc} K_1(\Lambda(G_\infty)) & \to & K_1(\mathcal{Q}(G_\infty)) & \stackrel{\partial}{\to} & K_0T(\Lambda(G_\infty)) \\ & & \mathrm{Det}\downarrow & & \\ & & \mathrm{Hom}^*(R_l(G_\infty), \mathcal{Q}^{\mathrm{c}}(\Gamma_k)^\times) & & . \end{array}$$

The asterisk $*$ indicated Galois compatibility and a certain W-twist condition of the homomorphisms $R_l(G_\infty) \to \mathcal{Q}^{\mathrm{c}}(\Gamma_k)^\times$.

In $K_0T(\Lambda(G_\infty))$ we have the element $\mho \stackrel{\mathrm{def}}{=} \tilde{\mho}_S$ refining the Iwasawa module X_∞; in the Hom group we have the function $L_k \stackrel{\mathrm{def}}{=} L_{k,S}$ arising from the l-adic L-functions. We know that there exists an $x \in K_1(\mathcal{Q}(G_\infty))$ with $\partial(x) = \mho$, *and with it the "main conjecture" of equivariant Iwasawa theory, up to the uniqueness claim, can be restated as*

$$\mathrm{Det}\,(x){L_k}^{-1} \in \mathrm{Det}\,K_1(\Lambda(G_\infty))\,.$$

Indeed, if $v \in K_1(\Lambda(G_\infty))$ has $\mathrm{Det}\,(v) = \mathrm{Det}\,(x){L_k}^{-1}$, then $L_k = \mathrm{Det}\,(xv^{-1})$ and $\partial(xv^{-1}) = \partial(x) = \mho$.

So we need to understand $\mathrm{Det}\,K_1(\Lambda(G_\infty))$.

Define Ψ on $\Lambda(\Gamma_k)$ to be the $\mathbb{Z}_l$-linear map sending γ to γ^l for $\gamma \in \Gamma_k$ (so, on identifying $\Lambda(\Gamma_k)$ with the ring $\mathbb{Z}_l[[T]]$ of power series, Ψ is the substitution $T \mapsto (1+T)^l - 1$ on power series). Ψ is naturally extended to $\Lambda^{\mathrm{c}}(\Gamma_k) = {\mathbb{Z}_l}^{\mathrm{c}} \otimes_{\mathbb{Z}_l} \Lambda(\Gamma_k)$.

Finally, ψ_l is the l^{th} Adams operation on $R_l(G_\infty)$, $\psi_l\chi(g) = \chi(g^l)$.

Lemma. $\mathrm{Det}\,K_1(\Lambda(G_\infty)) \subset \mathrm{HOM}(R_l(G_\infty), \Lambda^{\mathrm{c}}(\Gamma_k)^\times) \subset \mathrm{Hom}^*(R_l(G_\infty), \Lambda^{\mathrm{c}}(\Gamma_k)^\times)$.

Above, HOM *means an extension of the* $*$*-condition, namely* $f(\chi)^l \equiv \Psi(f(\psi_l\chi)) \bmod l$ for $f \in \mathrm{Hom}^*$.

¶the symbol $\boxed{\star}$ in the margin refers to notation or results from A. Weiss' talk (compare also our preprint "Toward equivariant Iwasawa theory, II", to appear in Indagationes Mathematicae [or see www.ritter-maths.de])

Note that $\mathrm{Det}\, K_1(\Lambda(G_\infty)) \subset \mathrm{Hom}^*(R_l(G_\infty), \Lambda^{c}(\Gamma_k)^\times)$ is an integrality statement. It follows immediately from the definition of Det since the characteristic polynomial of a unit $u \in \Lambda(G_\infty)^\times$ acting on the the $\mathcal{Q}^c(G_\infty)$-module $\mathfrak{V}_\chi = \mathrm{Hom}_{\mathbb{Q}_l{}^c[H]}(V_\chi, \mathcal{Q}^c(G_\infty))$, which is induced by the irreducible character χ, has coefficients in $\Lambda^c(\Gamma_k)$. I should perhaps mention here that the canonical map $\Lambda(G_\infty)^\times \to K_1(\Lambda(G_\infty))$ is surjective, since $\Lambda(G_\infty)$ is a semilocal ring.

The congruence yielding HOM is first checked for abelian characters of G_∞ and then, using explicit Brauer induction, extended to nonabelian G_∞. The analogous computation, but for finite rather than profinite groups, is already contained in Snaith's book *Explicit Brauer Induction*. With us, it is based on the formula

$$\chi \text{ abelian} \quad \Longrightarrow \quad (\mathrm{Det}\, g)(\chi) = \chi(g)\overline{g} \quad \text{for } g \in G_\infty \text{ with image } \overline{g} \text{ in } \Gamma_k\,,$$

the proof of which I omit.

Now, choose a central open $(\mathbb{Z}_l \simeq)\, \Gamma \leq G_\infty$ contained in the kernel of χ and take a coset decomposition $G_\infty = \dot{\bigcup}_{t\in T} t\Gamma$. Let $v = \sum_{t\in T} v_t t$, with $v_t \in \Lambda(\Gamma)$, be $\in \Lambda(G_\infty)^\times \twoheadrightarrow K_1(\Lambda(G_\infty))$. So $(\mathrm{Det}\, v)(\chi) = \sum_{t\in T} \overline{v}_t \chi(t)\overline{t}$ and $(\mathrm{Det}\, v)(\chi)^l \equiv \sum_{t\in T} \overline{v}_t^l \chi(t)^l \overline{t}^l \bmod l\,.$

On the other hand, $\Psi(\mathrm{Det}\,(v)(\psi_l\chi)) = \sum_{t\in T} \overline{\Psi(v)}\chi(t^l)\overline{t}^l = \sum_{t\in T} \Psi(\overline{v})\chi(t^l)\overline{t}^l$, so the congruence $\Psi(\overline{v}_t) \equiv \overline{v}_t^l \bmod l$ finishes the proof.

I leave it by that; how to apply explicit Brauer induction in order to cover the nonabelian characters of G_∞ as well we will see in a minute in a different context.

Having arrived at this stage, we remember from the abelian case that then there was a unique $\Theta \overset{\text{def}}{=} \tilde{\Theta}_S$ in $K_1(\mathcal{Q}(G_\infty))$ with $\mathrm{Det}\,(\Theta) = L_k$. This Θ arises from applying the canonical surjection $\mathcal{Q}(G_\infty)^\times \twoheadrightarrow K_1(\mathcal{Q}(G_\infty))$ to *Serre's pseudomeasure* $\lambda \in \mathcal{Q}(G_\infty)^\times$, which we saw in Al's talk. And identifying $\Lambda(G_\infty)$ and $\mathbb{Z}_l[H][[T]]$ on account of $G_\infty = H \times \Gamma$, $\Gamma \simeq \Gamma_k$, this λ equals T^{-1} times a power series in T with coefficients in $\mathbb{Z}_l[H]$.

With $G = G_\infty, \Gamma_k, \ldots$, we write $\Lambda(G)_\bullet$ for the ring of fractions of $\Lambda(G)$ with denominators which are not zero divisors in $\Lambda(G)/l\Lambda(G)$. We also set $\Lambda^c_\bullet(\Gamma_k) = \mathbb{Z}_l{}^c \otimes_{\mathbb{Z}_l} \Lambda(\Gamma_k)_\bullet$. Since $\Lambda(\Gamma_k)_\bullet$ is a discrete valuation ring, the above lemma extends to the $\bullet$-situation.

Recall at this point that when G_∞ is abelian, so when there exists Serre's pseudomeasure λ, it is actually contained in $\Lambda(G_\infty)_\bullet \cap \mathcal{Q}(G_\infty)^\times$.

Theorem. $\quad L_k \in \mathrm{HOM}(R_l(G_\infty), \Lambda^c_\bullet(\Gamma_k)^\times)$

For the proof we first assume that G_∞ is abelian, so $G_\infty = \Gamma \times H$ with a lift Γ of Γ_k in G_∞. We recall that λ is in $\Lambda(G_\infty)_\bullet \cap \mathcal{Q}(G_\infty)^\times$ and maps to Θ. If already $\lambda \in \Lambda(G_\infty)_\bullet{}^\times$, then $L_k = \mathrm{Det}\,(\tilde{\lambda})$ with $\tilde{\lambda} =$ image of λ in $K_1(\Lambda(G_\infty)_\bullet)$ and the lemma finishes the proof. In order to see that $\lambda \in \Lambda(G_\infty)_\bullet{}^\times$ we compute the determinant of the action of λ on the free

$\Lambda(\Gamma)_\bullet$-module $\Lambda(G_\infty)_\bullet$:

$$\begin{aligned}
&\det_{\Lambda(\Gamma)_\bullet}(\lambda \mid \Lambda(G_\infty)_\bullet) = \det_{\mathcal{Q}^c(\Gamma)}(\lambda \mid \mathcal{Q}^c(G_\infty)) \\
&= \det_{\mathcal{Q}^c(\Gamma_k)}(\lambda \mid \mathcal{Q}^c(\Gamma_k) \otimes_{\mathcal{Q}^c(\Gamma)} \mathrm{Hom}_{\mathbb{Q}_l{}^c}(\mathbb{Q}_l{}^c, \mathcal{Q}^c(G_\infty))) \\
&= \det_{\mathcal{Q}^c(\Gamma_k)}(\lambda \mid \mathrm{Hom}_{\mathbb{Q}_l{}^c[H]}(\mathrm{ind}_\Gamma^{G_\infty}(\mathbb{Q}_l{}^c), \mathcal{Q}^c(G_\infty))) = (\mathrm{Det}\,\Theta)(\mathrm{ind}_\Gamma^{G_\infty}(1)) \\
&= L_k(\mathrm{ind}_\Gamma^{G_\infty}(1)) = L_{k'}(1) = \tfrac{G_{1,S}(\gamma-1)}{H_1(\gamma-1)}
\end{aligned}$$

where $k' = K_\infty^\Gamma$ and γ is the topological generator of Γ used in the definition of the Deligne-Ribet Iwasawa power series G and H. And since $\mu = 0$, $\frac{G_{1,S}(\gamma-1)}{H_1(\gamma-1)}$ is indeed a unit in $\Lambda(\Gamma_{k'})_\bullet$. So the abelian case is settled.

For general G_∞ we employ again explicit Brauer induction to get both

$$\chi = \sum_i n_i \mathrm{ind}_{G_i'}^{G_\infty}(\chi_i') \quad \text{and} \quad \psi_l\chi = \sum_i n_i \mathrm{ind}_{G_i'}^{G_\infty}(\psi_l\chi_i')\,,$$

with suitable open subgroups G_i' of G_∞ and characters χ_i' of G_i' with open kernel. Since L_k is compatible with induction and inflation of characters, we get, with $k_i' = K_\infty^{G_i'}$,

$$\begin{aligned}
&L_k(\chi)^l = \textstyle\prod_i L_{k_i'}(\chi_i')^{ln_i} \equiv \prod_i \Psi\Big(L_{k_i'}(\psi_l\chi_i')\Big)^{n_i} \\
&= \textstyle\prod_i \Psi\Big(L_k(\mathrm{ind}_{G_i'}^{G_\infty}(\psi_l\chi_i'))\Big)^{n_i} = \Psi(L_k(\psi_l\chi)) \bmod l\Lambda_\bullet^c(\Gamma_k)\,.
\end{aligned}$$

This proves the theorem.

The abelian model now suggests to replace the question

$$\mathrm{Det}\,(x)L_k{}^{-1} \overset{?}{\in} \mathrm{Det}\,K_1(\Lambda(G_\infty)) \quad \text{by} \quad L_k \overset{?}{\in} \mathrm{Det}\,K_1(\Lambda(G_\infty)_\bullet)\,.$$

A preimage of L_k in $K_1(\Lambda(G_\infty)_\bullet)$, in analogy to Serre's $\lambda \in \Lambda(G_\infty)_\bullet{}^\times$ in the abelian case, should be called *a nonabelian pseudomeasure.* Note that x does not show in the second question anymore, so there is no reference to $\mho$ left. $\boxed{\star}$

For l-groups G_∞ the two questions above are indeed equivalent. I show how the second implies the first one.

Let M_∞ be the maximal abelian l-extension of K_∞ (unramified off S) and let X_∞ be the Galois group of M_∞/K_∞.

By $\mu = 0$, $(X_\infty)_\bullet = 0$. Namely, choose a splitting $G_\infty = H \rtimes \Gamma$ and let k' be the fixed field of Γ. Then the cyclotomic l-extension $(k')_\infty$ of k' equals K_∞, hence X_∞ has μ-invariant 0 as $\Lambda(\Gamma)$-module, i.e., the l-power torsion submodule of X_∞ is finite. Thus the $\Lambda(G_\infty)_\bullet$-torsion module $(X_\infty)_\bullet$ is finite since $\Lambda(G_\infty)_\bullet$ is a *dvr* with its maximal ideal generated by l. But then $(X_\infty)_\bullet = 0$, because the residue field of $\Lambda(G_\infty)_\bullet$ is infinite.

Recall now the definition of $\mho$ from the previous talk: it results from the commuting diagram on the right and is defined by $\mho = [\operatorname{coker} \varPsi] - [\operatorname{coker} \tilde{\psi}]$. The localized diagram induces, by $(X_\infty)_\bullet = 0$ and $(\mathbb{Z}_l)_\bullet = 0$, an isomorphism $(\operatorname{coker} \varPsi)_\bullet \simeq (\operatorname{coker} \tilde{\psi})_\bullet$. Note that $(\mathbb{Z}_l)_\bullet = 0$ as γ acts trivially and $T = \gamma - 1 \in \Lambda(\Gamma)_\bullet^\times$. Consequently, $\mho_\bullet = 0$.

$$\begin{tikzcd} & \Lambda(G_\infty) \arrow[r, phantom, "="] \arrow[d, hook, "\varPsi"'] & \Lambda(G_\infty) \arrow[d, hook, "\tilde{\psi}"'] & \\ X_\infty \arrow[r, tail] & Y_\infty \arrow[r] \arrow[d] & \Lambda(G_\infty) \arrow[r, two heads] \arrow[d] & \mathbb{Z}_l \\ & \operatorname{coker} \varPsi \arrow[r] & \operatorname{coker} \tilde{\psi} & \end{tikzcd}$$

We know that $\exists\, x \in K_1(\mathcal{Q}(G_\infty))$ with $\partial(x) = \mho$, whence $\partial_\bullet : K_1(\mathcal{Q}(G_\infty)) \to K_0T(\Lambda(G_\infty)_\bullet)$ takes x to zero. Thus $x \in \operatorname{im}(K_1(\Lambda(G_\infty)_\bullet) \to K_1(\mathcal{Q}(G_\infty)))$ and $\operatorname{Det} x \in \operatorname{Det} K_1(\Lambda(G_\infty)_\bullet)$. The hypothesis $L_k \in \operatorname{Det} K_1(\Lambda(G_\infty)_\bullet)$ combined with the above says that $(\operatorname{Det} x)L_k^{-1}$ is in $\operatorname{Det} K_1(\Lambda(G_\infty)_\bullet)$. But it belongs also to $\operatorname{Hom}^*(R_l(G_\infty), \Lambda^{\mathrm{c}}(\Gamma_k)^\times)$, as was shown in the last talk. So we are done once we have the containment $\boxed{\star}$

$$(\#) \qquad \operatorname{Det} K_1(\Lambda(G_\infty)_\bullet) \cap \operatorname{Hom}^*(R_l(G_\infty), \Lambda^{\mathrm{c}}(\Gamma_k)^\times) \overset{!}{\subseteq} \operatorname{Det} K_1(\Lambda(G_\infty)),$$

with the intersection read in $\operatorname{Hom}^*(R_l(G_\infty), \mathcal{Q}^{\mathrm{c}}(\Gamma_k)^\times)$.

For l-groupos G_∞ we employ logarithmic methods to get (#). Logarithmic methods were first applied in the discussion of additive Galois structures, and indeed, the Taylor-Oliver integral group logarithm is the key tool when calculating determinants in Fröhlich's work on rings of integers. A generalization to the profinite case is however not straightforward. We would need a logarithm which is defined on $K_1(\Lambda(G_\infty)_\bullet)$ (or on $\Lambda(G_\infty)_\bullet{}^\times$) and takes values in $\Lambda(G_\infty)_\bullet/[\Lambda(G_\infty)_\bullet, \Lambda(G_\infty)_\bullet]$. But $\Lambda(G_\infty)_\bullet$ is not l-complete, so we run into difficulties with convergence and are forced to first complete $\Lambda(G_\infty)$ at l. This then right away requires to proving (#) with the localization replaced by the completion.

The logarithm LOG takes us to the additive world where we now need to consider LOG images y of elements x in the intersection (#), that is, elements in $y \in \Lambda(G_\infty)/[\Lambda(G_\infty), \Lambda(G_\infty)]$. The abelian case allows us to assume that the function $\operatorname{Tr}(y)$ vanishes when applied to the abelian characters of G_∞. Thus y belongs to the kernel of $\Lambda(G_\infty)/[\Lambda(G_\infty), \Lambda(G_\infty)] \to \Lambda(G_\infty^{\mathrm{ab}})$ and this kernel equals $\operatorname{LOG}\Big(1 + \ker(\Lambda(G_\infty) \to \Lambda(G_\infty^{\mathrm{ab}}))\Big)$. We may consequently assume that $y = 0$ and then $\operatorname{LOG} \circ \operatorname{Det} = \operatorname{Tr} \circ \operatorname{LOG}$ and the congruences for Dets give, first, $(\operatorname{Det} x)(\chi)^l = \Psi(\operatorname{Det} x)(\psi_l \chi)$ for the preimage x of y and all χ and then, by iterating, $(\operatorname{Det} x)^{l^m} = 1$ (some m). But now $\operatorname{Det} x = 1$ by a generalization of Wall's theorem on torsion units, and we are done since we have changed things only by $1 + \ker\Big(\Lambda(G_\infty) \to \Lambda(G_\infty^{\mathrm{ab}})\Big) \subset \Lambda(G_\infty)^\times$.

I better stop here, but before it let me summarize where we are with the "main conjecture" of equivariant Iwasawa theory:

> *Assume that G_∞ is an l-group. If L_k is in $\operatorname{Det} K_1(\Lambda(G_\infty)_\bullet)$, i.e., if there exists a nonabelian pseudomeasure, then the "main conjecture" holds (up to uniqueness of Θ). The required containment holds for abelian G_∞.*