



- $l \neq 2$, all number fields totally real
- K_{∞}/k Galois with H finite and k_{∞}/k the cyclotomic l -extension. So $G_{\infty} = G(K_{\infty}/k)$ has $H \hookrightarrow G_{\infty} \rightarrow T_k$.
- S finite set of primes of k containing l, ∞ and all which ramify in K_{∞} . Let M_{∞} be the max. abelian l -extension of K_{∞} which is unramified outside S . $X_{\infty} = G(M_{\infty}/K_{\infty})$ hence $X_{\infty} \hookrightarrow G(M_{\infty}/k) \rightarrow G_{\infty}$.

- $\Lambda(G) = \mathbb{Z}_l[[G]]$
 $Q(G) = \text{total ring of fractions of } \Lambda(G)$
 $K_1(\Lambda(G)) \xrightarrow{\partial} KT(\Lambda(G)) \rightarrow K_0(\Lambda(G))$
 $K_1(\Lambda(G)) \rightarrow K_1(Q(G)) \xrightarrow{\partial} K_0(Q(G))$
 the localization sequence
 Fix \mathbb{Q}_l^c an algebraic closure of \mathbb{Q}_l ,
 put $\mathbb{Z}_l^c = \text{integral closure of } \mathbb{Z}_l \text{ in } \mathbb{Q}_l^c$.
 $\Lambda^c(G) = \mathbb{Z}_l^c \otimes \Lambda(G)$, $Q^c(G) = \mathbb{Q}_l^c \otimes Q(G)$
- $R_l(G) = \mathbb{Z}$ -span of characters of f.d.
 \mathbb{Q}_l^c -representations of G with open kernel

The algebraic invariant From the group 2 extension $X_\infty \hookrightarrow G(M_\infty/\mathbb{Q}) \rightarrow G_\infty$ we get a natural exact $\Lambda(G_\infty)$ -module sequence

$$0 \rightarrow X_\infty \rightarrow Y_\infty \rightarrow \Lambda(G_\infty) \rightarrow \mathbb{Z}_\ell \rightarrow 0$$

Prop. (NQ-D) $\text{pd}_{\Lambda(G_\infty)} Y_\infty \leq 1$.

But Y_∞ is not a torsion $\Lambda(G_\infty)$ -module so to get into $K_0 T(\Lambda(G_\infty))$ choose monomorphism $\psi: \Lambda(G_\infty) \rightarrow \Delta(G_\infty) = \ker(\Lambda(G_\infty) \rightarrow \mathbb{Z})$ and factor it through Y_∞ to get $\text{aug } \psi$

$$\begin{array}{ccc} \Lambda(G_\infty) & = & \Lambda(G_\infty) \\ \Psi \downarrow & & \downarrow \psi \end{array}$$

$$X_\infty \longrightarrow Y_\infty \longrightarrow \Lambda(G_\infty) \longrightarrow \mathbb{Z}_\ell$$

Def $U_S = [\text{coker } \Psi] - [\text{coker } \psi]$ in $K_0 T(\Lambda(G_\infty))$.

This class, which is independent of the choice of ψ and Ψ , is now the main algebraic object.

The goal now is to specify $\Theta_S \in K_1(Q(G_\infty))$ in terms of ℓ -adic Artin L-functions so that $\partial(\Theta_S) = U_S$.

$K_1(Q(G_\infty))$ If χ is the character of the \mathbb{Q}_ℓ^c -representation space V_χ of G_∞ then

$$W_\chi = \text{Hom}_{\mathbb{Q}_\ell^c[H]}(V_\chi, \mathbb{Q}_\ell^c(G_\infty))$$

is a $\mathbb{Q}_\ell^c(\Gamma_k)$ - $\mathbb{Q}_\ell^c(G_\infty)$ bimodule with $\mathbb{Q}_\ell^c(G_\infty)$ acting via its right action on itself

and $Q^c(\Gamma_k)$ acting "diagonally", i.e. induced by $(\gamma f)(v) = \gamma(f(\gamma^{-1}v))$ for $\gamma \in G_\infty$ with image $\gamma_k \in \Gamma_k$. 3

Prop. There's a homomorphism

$\text{Det} : K_1(Q(G_\infty)) \rightarrow \text{Hom}^*(R_\ell(G_\infty), Q^c(\Gamma_k)^\times)$ which takes the generators $[Q(G_\infty), q]$, with $q \in Q(G_\infty)^\times$, to

$$(\text{Det } q)(X) = \det_{Q^c(\Gamma_k)}(q/W_X).$$

Here * means $f : R_\ell(G_\infty) \xrightarrow{k} Q^c(\Gamma_k)^\times$ is restricted to be

a) $G(\mathbb{Q}_\ell^\times/\mathbb{Q}_\ell)$ -equivariant

b) compatible with W-twists i.e.

$$f(X \otimes \rho) = \rho^\#(f(X)) \text{ for all } X \in R_\ell(G_\infty)$$

whenever $\rho : G_\infty \rightarrow (\mathbb{Q}_\ell^\times)^\times$ has $\rho(H) = 1$.

Here $\rho^\#$ is the \mathbb{Q}_ℓ^\times -automorphism of $Q^c(\Gamma_k)$ so that $\rho^\#(\gamma_k) = \rho(\gamma_k) \otimes \gamma_k$, $\gamma \in \Gamma_k$.

"Proof" If X is irreducible we get a ring homomorphism $j_X : \text{cent}(Q^c(G_\infty)) \rightarrow Q^c(\Gamma_k)$ by $w c = j_X(c) w$ for all $w \in W_X$. The

resulting right $Q^c(\Gamma_k) \otimes_{\substack{\kappa \in \text{cent}(Q^c(G_\infty))}} Q^c(G_\infty) -$

module W_X is irreducible. In this

the Wedderburn components of $Q^c(G_\infty)$ are in bijection with the W-twist classes of irreducible \mathbb{Q}_ℓ^\times -characters of G_∞ .

Taking fixed points under $G(\mathbb{Q}_\ell^\times/\mathbb{Q}_\ell)$ gets $K_1(Q(G_\infty))$.

Corollary

4

$$\begin{array}{ccc} K_1(Q(G_{\infty})) & & \\ \searrow \text{nr} & & \swarrow \text{Dot} \\ \text{cent}(Q(G_{\infty}))^{\times} & \xrightarrow{\sim} & \text{Hom}^*(R_{\ell}(G_{\infty}), Q^c(\Gamma_{\ell})^{\times}) \end{array}$$

The analytic invariant Let $L_{l,S}(s, \chi)$ be the S -truncated ℓ -adic L -function associated to $\chi \in R_{\ell}(G_{\infty})$, i.e. with all Euler factors at primes in S removed before ℓ -adic interpolation. According to Deligne-Ribet etc./Greenberg/Wiles these are Iwasawa functions for irreducible χ i.e. once a generator γ_k of Γ_{ℓ}^c is fixed, then there is a $G_{\chi,S}(T) \in \mathbb{Z}_{\ell}^c \otimes \mathbb{Z}_{\ell}[[T]]$ so that $L_{l,S}(1-n, \chi) = \frac{G_{\chi,S}(u^{n-1})}{H_{\chi}(u^{n-1})} \text{ in } \mathbb{Z}$.

Here $H_{\chi}(T) = \begin{cases} \chi(\gamma_k)(1+T) - 1, & H \subseteq \ker \chi \\ 1, & \text{else} \end{cases}$

and $u \in 1 + \ell \mathbb{Z}_{\ell}$ so $\gamma_{\ell^{\infty}}^{\gamma_k} = \gamma_{\ell^{\infty}}^u$.

Def $L_k = L_{k,S} \in \text{Hom}^*(R_{\ell}(G_{\infty}), Q^c(\Gamma_{\ell})^{\times})$ is given by $L_k(\chi) = \frac{G_{\chi,S}(\gamma_k - 1)}{H_{\chi}(\gamma_k - 1)}$

This is independent of the choice of γ_k and in Hom^* by standard properties.

Equivariant Main Conjecture"

$$K_1(Q(G_\infty)) \xrightarrow{\partial} K_0 T(\Lambda(G_\infty))$$

↓ Det ↓ \cup_S

$$\text{Hom}^*(R_\ell(G_\infty), Q^c(\Gamma_k^\circ)^\times) \ni L_k$$

Theorem

$\exists x \in K_1(Q(G_\infty))$ so $\partial x = \cup_S$. Every such x has $(\text{Det } x)L_k^{-1} \in \text{Hom}^*(R_\ell(G_\infty), \Lambda^c(\Gamma_k^\circ)^\times)$

"Proof": The existence of x follows from \cup restricting (to open subgroups) and deflating (modulo finite normal subgroups) to \cup . Since L_k does the same, the rest of the assertion also reduces to the case of abelian G_∞ .

For each proper character χ of G_∞ there is a commutative square

$$\begin{array}{ccc} K_1(Q(G_\infty)) & \xrightarrow{\partial} & K_0 T(\Lambda(G_\infty)) \\ \chi_* \downarrow & & \downarrow \chi_* \\ K_1(Q^c(\Gamma_k^\circ)) & \xrightarrow{\partial} & K_0 T(\Lambda^c(\Gamma_k^\circ)) \end{array}$$

with vertical maps induced by

$$\text{Hom}_{\mathbb{Q}_\ell[H]}^c(V_\chi, \mathbb{Q}_\ell^c \otimes -), \text{Hom}_{\mathbb{Z}_\ell[H]}^c(M_\chi, \mathbb{Z}_\ell^c \otimes -)$$

respectively, where M_χ is a \mathbb{Z}_ℓ^c -lattice on V_χ which is stable under the

Conjecture 5
 There is a unique Θ_S in $K_1(Q(G_\infty))$ with $\text{Det } \Theta_S = L_k$. This has $\partial \Theta_S = \cup_S$.

G_∞ -action. These maps do not depend on the choice of M_X on V_X .

The left map has $\det \circ \chi_* = \text{Det}(\chi)$ where $\det : K_1(Q^c(\Gamma_k)) \xrightarrow{\cong} Q^c(\Gamma_k)^\times$. The right map has the crucial property $\chi_*(U_S) = [\text{Hom}_{Z_l^c[H]}(M_X, Z_l^c \otimes_{Z_l} X_\infty)] - [\text{Hom}_{Z_l^c[H]}(M_X, Z_l^c \otimes_{Z_l} Z_l)]$.

$$\begin{array}{ccc} \chi_*(x) & \in & K_1(Q^c(\Gamma_k)) \longrightarrow K_0 T(\Lambda^c(\Gamma_k)) \\ \downarrow \det \quad \downarrow \simeq & & \downarrow \psi \\ (\text{Det } x)(x) & \in & Q^c(\Gamma_k)^\times \\ \text{we get to } \chi_*(U_S) & \leftrightarrow & (\text{Det } x)(x) \bmod \Lambda^c(\Gamma_k)^\times \end{array}$$

Fix a generator γ_k of Γ_k and $\gamma \in G_\infty$ mapping to it by $G_\infty \rightarrow \Gamma_k$. With G_∞ abelian, every irreducible X can be γ -twisted to arrange $X(\gamma) = 1$. As $\rho^\#$ preserves $\Lambda^c(\Gamma_k)^\times$ we need to show $(\text{Det } x)(x) L_k(x)^{-1} \in \Lambda^c(\Gamma_k)^\times$ when $X(\gamma) = 1$.

Since $[\text{Hom}_{Z_l^c[H]}(M_X, Z_l^c)] \leftrightarrow H_X(\gamma_k - 1) \bmod \Lambda^c(\Gamma_k)^\times$ trivially, this is precisely the situation in which Wiles' results give $[\text{Hom}_{Z_l^c[H]}(M_X, Z_l^c \otimes_{Z_l} X_\infty)] \leftrightarrow G_{X,S}(\gamma_k - 1) \Lambda^c(\Gamma_k)^\times$.

Abelian revision Suppose G_∞ abelian. Let $G_S = G(k_S/k)$ where k_S is the max. abelian extension of k unramified off S .
 $\gamma_S \in G_S$ Then $k_\infty \subseteq k_S$ so fixing a generator γ_k of Γ_k we lift it to $\gamma \in G_\infty$ and then to $\gamma_k \in \Gamma_k$. Now the pseudo measure λ_S associated to k_S/k by Serre has

$(\gamma_S - 1)\lambda_S \in \mathbb{Z}_\ell[[G_S]]$ and specializes to $L_{k,S}(1-n, \chi)$ for $n \geq 1$ and characters χ of G_S (by evaluating $\chi \chi_{\text{cycl}}^n$ on λ_S).

Define $\lambda_\infty \in Q(G_\infty)^\times$ by setting $(\gamma - 1)\lambda_\infty = \text{image of } (\gamma_S - 1)\lambda_S$ under $\mathbb{Z}_\ell[[G_S]] \rightarrow \mathbb{Z}_\ell[[G_\infty]]$. Then also

$$\det \lambda_\infty = L_{k,S}$$

i.e. Θ_S is the pseudomeasure associated to k_∞/k .