ASYMPTOTIC COHOMOLOGY OF CIRCULAR UNITS

by

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Abstract. — Let F be a number field, abelian over $\mathbb{Q}$, and fix a prime $p \neq 2$. Consider the cyclotomic $\mathbb{Z}_p$-extension F_∞/F and denote F_n the n^{th} finite subfield and C_n its group of circular units. Then the Galois groups $G_{m,n} = \text{Gal}(F_m/F_n)$ act naturally on the C_m's (for any $m \geq n >> 0$). We compute the Tate cohomology groups $\widehat{H}^i(G_{m,n}, C_m)$ for $i = -1, 0$ without assuming anything else neither on F nor on p.

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Introduction

Let F be a number field, abelian over $\mathbb{Q}$, and fix a prime $p \neq 2$. Consider the cyclotomic $\mathbb{Z}_p$-extension F_∞/F and denote F_n its n^{th} finite layer, i.e. F_n/F is the unique sub-extension of degree p^n of F_∞/F. Following Sinnott ([**S**]) we define, for every F_n, its group of circular units C_n. Then the cyclic Galois groups $G_{m,n} = \text{Gal}(F_m/F_n)$ act naturally on the C_m's (for any $m \geq n \geq 0$). The present note aims to compute the Tate cohomology groups $\widehat{H}^i(G_{m,n}, C_m)$ for $i = -1, 0$ without assuming anything else on F. To complete our computations in this general case, we need to take into account differences between Sinnott's and Washington's versions of circular units, even at the infinite level. In the process, as it does not involve any other difficulties, we compute the cohomology of this second version, and also the $\Gamma_n = \text{Gal}(F_\infty/F_n)$-cohomology of both inductive limits. Unfortunately we only

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succeed in describing these cohomology groups by assuming that n is not less than some natural number N (see lemma 2.8, lemma 3.1 and theorem 3.5), hence the word "asymptotic" in the title. On the other hand if we do assume *ad hoc* hypotheses on F, then the natural number N may be chosen to be equal to 0, and our general approach recovers all previously known special cases (see e.g. articles [**K1, K2, K3, KO**], and §III.4 of [**B1**], and §4 of [**N**]).

1. Two versions of cyclotomic units

We start by recalling the definitions of Sinnott's and Washington's circular units. Fix an embedding of $\overline{\mathbb{Q}}$ in $\mathbb{C}$ and denote $\zeta_m = \exp(2i\pi/m)$.

Definition 1.1. — Let F be an abelian number field and f be its conductor, that is the smallest integer such that $F \subset \mathbb{Q}(\zeta_f)$. The group of units of F will be denoted $U(F)$.

1. We call group of circular numbers of F and denote $\mathrm{C}yc(F)$ the Galois submodule of $F^\times$ generated by -1 and the numbers $N_{\mathbb{Q}(\zeta_d)/\mathbb{Q}(\zeta_d)\cap F}(1-\zeta_d)$, where d runs through all divisors of f greater than 1.
2. We call Sinnott's group of circular units of F (see [**S**]) and denote $C(F)$ the intersection
$$C(F) = \mathrm{Cyc}(F) \bigcap U(F).$$
3. We call Washington's group of circular units of F (see p.143 of [**W**]) and denote $W(F)$ the intersection
$$W(F) = \mathrm{Cyc}(\mathbb{Q}(\zeta_f)) \bigcap U(F).$$

Fix now a prime number $p \neq 2$ and consider the cyclotomic $\mathbb{Z}_p$-extension F_∞/F of any abelian number field F. Then all finite layers F_n are still abelian over $\mathbb{Q}$ and the notations $C_n = C(F_n)$ and $W_n = W(F_n)$ make sense. We also abbreviate $U(F_n)$ to U_n. When necessary we will rather consider the pro-p-completions $\overline{C}(F) = C(F)\otimes\mathbb{Z}_p$, $\overline{W}_n = W_n\otimes\mathbb{Z}_p$, $\overline{C}_n = C_n\otimes\mathbb{Z}_p$ and $\overline{U}_n = U_n\otimes\mathbb{Z}_p$. We put as usual $\Gamma = \mathrm{Gal}(F_\infty/F)$ and denote $\Lambda = \mathbb{Z}_p[[\Gamma]]$ for the Iwasawa algebra. For all n we have obvious inclusions $C_n \subset W_n$. Counterexamples to the equality are known. Moreover there exist also abelian number fields F such that even the projective limits $\overline{C}_\infty = \varprojlim \overline{C}_n$ and $\overline{W}_\infty = \varprojlim \overline{W}_n$ with respect to norm maps along F_∞/F disagree (see e.g. [**B2**]; other examples were built simultaneously and independently by Kučera in [**Kuč**]).

Definition 1.2. — Let us denote KN_∞ the quotient module $KN_\infty = \overline{W}_\infty/\overline{C}_\infty$. This module has been defined as a capitulation kernel for unit classes and called "Kučera–Nekovář kernel" in [**BN**] and [**NL**].

Let $\overline{U}_\infty$ be the projective limit of the $\overline{U}_n$. Let $\overline{\mu}_n$ be the p^{th}-power roots of unity in F_n and $\overline{\mu}_\infty$ be their projective limit. Let F^+ be the maximal real subfield of F, and for any p-adic $\mathrm{Gal}(F/F^+)$-module M let us abbreviate $M^{\mathrm{Gal}(F/F^+)}$ to M^+. We have $\overline{\mu}_\infty \subset \overline{C}_\infty \subset \overline{W}_\infty \subset \overline{U}_\infty$ and with M standing for any of these four modules of units we have a direct sum decomposition $M = M^+ \oplus \overline{\mu}_\infty$. For later use, we summarize some other known properties of these modules in a proposition.

Proposition 1.3. — *Let $r = [F^+ : \mathbb{Q}]$.*

1. *The module KN_∞ is finite.*
2. *The module $\overline{W}^+_\infty$ is Λ-free of rank r.*
3. *The Λ-module $\overline{C}^+_\infty$ is of rank r.*
4. *The module $\overline{C}^+_\infty$ is Λ-free if and only if $KN_\infty = 0$.*
5. *The module KN_∞ is the maximal finite submodule of $\overline{U}_\infty/\overline{C}_\infty = \overline{U}^+_\infty/\overline{C}^+_\infty$.*

Proof. — The indices $(W_n : C_n)$ are bounded uniformly in n, this is the main result of [**KN**]. Assertion 1 follows. Assertion 2 is a theorem of Kuz′min (see [**Kuz2**] or discussion before proposition 3.6 of [**B2**]). Assertion 3 is a consequence of 1 and 2. Assertion 4 is proposition 3.6 of [**B2**] (see also [**BN**], proposition 2.3 (*ii*)). Let us denote MF_∞ for the maximal finite submodule of $\overline{U}_\infty/\overline{C}_\infty$. By assertion 1 we already have an inclusion $KN_\infty \subset MF_\infty$. But since both $\overline{U}^+_\infty$ and $\overline{W}^+_\infty$ are Λ-free the quotient $\overline{U}_\infty/\overline{W}_\infty$ has no non-trivial finite submodule. This shows assertion 5. □

In [**BN**] (proposition 2.3 (*ii*)) the module denoted KN_F was shown to be equal to MF_∞, so that notations are coherent. For our present purpose we will make no use of the interpretation in terms of capitulation kernels. In the sequel we will consider this finite module KN_∞ as a parameter depending on the base field F. As written above, two different infinite families of couples (F, p) giving non-trivial KN_∞ are described in [**Kuč**] and [**B2**]. On the other hand the definition 6.4, together with theorem 6.6 in the appendix to [**NL**] gives a criterion for the triviality of KN_∞. Up to now all known examples of trivial KN_∞ satisfy this criterion.

2. Universal norms of circular units

To compute the cohomology of C_m and W_m we first use the simpler structure of $\overline{W}^+_\infty$ and of $\overline{C}^+_\infty$ to recover the cohomology of the "universal norms" modules $\widetilde{C}^+_n \subset \overline{C}_n$ and $\widetilde{W}^+_n \subset \overline{W}_n$, which are defined as the images of the usual projection maps. Then in section 3 we will control the deviation between the initial modules and their universal norms.

Definition 2.1. — Let $n \in \mathbb{N}$.

1. Let $\widetilde{C}_n = \mathrm{Im}(\overline{C}_\infty \longrightarrow \overline{C}_n)$ be the module consisting of the universal norms of Sinnott's units.
2. Let $\widetilde{W}_n = \mathrm{Im}(\overline{W}_\infty \longrightarrow \overline{W}_n)$ be the module consisting of the universal norms of Washington's units.

By an usual compactness argument we have e.g. $\widetilde{C}_n = \bigcap_{m\geq n} N_{F_m/F_n}(\overline{C}_m)$ hence the terminology "universal norms".

In the sequel we will obtain asymptotic results but will try to collect as much information as possible about the first layer F_n from which our results apply. For the cohomology of $\widetilde{W}_n$, this is easier. It is well known that there exists some integer n such that F_∞/F_n is totally ramified at every place above p.

Definition 2.2. — We will denote n_d the smallest integer such that no place above p do splits anymore in F_∞/F_{n_d}.

If p is (at most) tamely ramified in $F/\mathbb{Q}$, then $n_d = 0$. For all $m \geq n \geq 0$ we will denote $\mathrm{Gal}(F_\infty/F_n)$ by Γ_n and $\mathrm{Gal}(F_m/F_n) \cong \Gamma_n/\Gamma_m$ by $G_{m,n}$. By the mere definition of Galois action on places, the group Γ_{n_d} acts trivially on the set S_∞ of places of F_∞ dividing p.

For later use we will isolate in a lemma here a precise statement for our context of the well known snake lemma :

Lemma 2.3. — *For all $m \in \mathbb{N}$ and all Λ-module M let us denote by M^{Γ_m} the Γ_m-invariant submodule of M and by M_{Γ_m} the Γ_m-coinvariant quotient module of M. Let $0 \to A \to B \xrightarrow{\varphi} C \to 0$ be an exact sequence of Λ-module. Pick for all m a topological generator γ_m of Γ_m. Then for all $m \in \mathbb{N}$ the map $\delta\colon C^{\Gamma_m} \longrightarrow A_{\Gamma_m}$ is well defined by the formula $\delta(\varphi(b)) = (\gamma_m - 1)b + (\gamma_m - 1)A$ and fits into the exact sequence*

$$0 \longrightarrow A^{\Gamma_m} \longrightarrow B^{\Gamma_m} \longrightarrow C^{\Gamma_m} \xrightarrow{\delta} A_{\Gamma_m} \longrightarrow B_{\Gamma_m} \longrightarrow C_{\Gamma_m} \longrightarrow 0.$$

Proof. — Just apply the snake lemma to the sequence $0 \to A \to B \xrightarrow{\varphi} C \to 0$ connected with itself by multiplication by $\gamma_m - 1$. □

Proposition 2.4. — *Let $s^+ = \#S^+_\infty$ be the number of places of F^+_∞ dividing p. Let n_d be as in definition 2.2. Then for all $m \geq n \geq n_d$, we have $\widehat{H}^0(G_{m,n}, \widetilde{W}_m) = 0$ and $H^1(G_{m,n}, \widetilde{W}_m)$ is free of rank s^+ over $\mathbb{Z}/p^{m-n}\mathbb{Z}$.*

Proof. — Fix $m \geq n \geq n_d$. The canonical surjection $\overline{W}_\infty \longrightarrow \widetilde{W}_m$ factors through $w_m\colon (\overline{W}_\infty)_{\Gamma_m} \twoheadrightarrow \widetilde{W}_m$. Set $T_m = \operatorname{Ker} w_m$. By proposition 1.3 we know that $(\overline{W}_\infty)_{\Gamma_m}$ is isomorphic to $\overline{\mu}_m \oplus (\overline{W}^+_\infty)_{\Gamma_m}$ with both summands $G_{m,0}$-cohomologically trivial. Therefore the cohomology of $\widetilde{W}_m$ is the cohomology of T_m shifted once. To conclude the proof of proposition 2.4 it suffices to show that T_m is a free $\mathbb{Z}_p$-module of rank s^+ with trivial action of $G_{m,n}$. For all n let $\overline{U}'_n$ denote the pro-p-completion of (p)-units of F_n, and $\overline{U}'_\infty$ be their projective limit. By two theorems of Kuz′min (see theorems 7.2 and 7.3 in [**Kuz1**]) the module $\overline{U}'_\infty$ is Λ-free and for all m the map $(\overline{U}'_\infty)_{\Gamma_m} \longrightarrow \overline{U}'_m$ is injective.

The sequence $0 \longrightarrow \overline{W}_\infty \longrightarrow \overline{U}'_\infty \longrightarrow \overline{U}'_\infty/\overline{W}_\infty \longrightarrow 0$ gives by lemma 2.3 $0 \longrightarrow (\overline{U}'_\infty/\overline{W}_\infty)^{\Gamma_m} \longrightarrow (\overline{W}_\infty)_{\Gamma_m} \longrightarrow (\overline{U}'_\infty)_{\Gamma_m}$ hence an isomorphism

$$T_m \cong (\overline{U}'_\infty/\overline{W}_\infty)^{\Gamma_m}.$$

This, together with the following lemma 2.5, completes the proof of the proposition 2.4. □

Lemma 2.5. — *Let s_m^+ be the cardinality of the set S_m^+ of places dividing p in the maximal real subfield F_m^+ of F_m. Note that for all $m \geq n_d$ the set S_m^+ is in bijection with S_∞^+.*

1. *The module $\overline{U}'_\infty/\overline{U}_\infty$ is $\mathbb{Z}_p$-free and pseudo-isomorphic to $\mathbb{Z}_p[S_\infty^+]$.*
2. *For all m the module $(\overline{U}'_\infty/\overline{W}_\infty)^{\Gamma_m}$ is $\mathbb{Z}_p$-free of rank s_m^+ with trivial action by Γ_{n_d}.*

Proof. — We first prove that 1 implies 2, then we will prove 1. Apply lemma 2.3 to the sequence

$$0 \longrightarrow \overline{U}_\infty/\overline{W}_\infty \longrightarrow \overline{U}'_\infty/\overline{W}_\infty \longrightarrow \overline{U}'_\infty/\overline{U}_\infty \longrightarrow 0.$$

We get

$$0 \longrightarrow (\overline{U}_\infty/\overline{W}_\infty)^{\Gamma_m} \longrightarrow (\overline{U}'_\infty/\overline{W}_\infty)^{\Gamma_m} \longrightarrow (\overline{U}'_\infty/\overline{U}_\infty)^{\Gamma_m} \longrightarrow (\overline{U}_\infty/\overline{W}_\infty)_{\Gamma_m}$$

By lemma 1.3 the module $\overline{U}_\infty/\overline{W}_\infty$ is pseudo-isomorphic to $\overline{U}_\infty/\overline{C}_\infty$, and as a consequence of Coleman's theory and Leopoldt's conjecture (see theorem 1.1 of [**B3**]) these two modules have finite Γ_m-invariants and co-invariants. But since $\overline{U}_\infty^+$ and $\overline{W}_\infty^+$ are Λ-free the quotient $\overline{U}_\infty/\overline{W}_\infty$ has no non-trivial finite submodule so that $(\overline{U}_\infty/\overline{W}_\infty)^{\Gamma_m} = 0$ and our previous sequence becomes

$$0 \longrightarrow (\overline{U}'_\infty/\overline{W}_\infty)^{\Gamma_m} \longrightarrow (\overline{U}'_\infty/\overline{U}_\infty)^{\Gamma_m} \longrightarrow \text{finite}.$$

Therefore assertion 2 follows from assertion 1. To prove 1 we use (normalized) valuations at places above p and consider the exact sequence :

$$0 \longrightarrow \overline{U}'_\infty/\overline{U}_\infty \xrightarrow{\text{val}} \mathbb{Z}_p[S_\infty] \longrightarrow D_\infty \longrightarrow 0,$$

where D_∞ stands for (the projective limit) of the p-part of the subgroup of ideal class groups generated by places above p. This shows already that $(\overline{U}'_\infty/\overline{U}_\infty)$ is $\mathbb{Z}_p$-free with trivial action of Γ_{n_d}. Proving that $(\overline{U}'_\infty/\overline{U}_\infty)$ is pseudo-isomorphic to $\mathbb{Z}_p[S_\infty^+]$ is then equivalent to prove that D_∞ is pseudo-isomorphic to $\mathbb{Z}_p[S_\infty]^- = \mathbb{Z}_p[S_\infty]/\mathbb{Z}_p[S_\infty^+]$. This last statement is a consequence of Leopoldt's conjecture (which holds true for our abelian field F) and is part of the folklore. For instance Greenberg in [**G**] §1 proves that D_∞^+ is finite and constructs in loc. cit. §2 a sequence of subgroups (denoted C_m in loc. cit. p.120) of uniformly bounded finite index in D_m, whose projective limit is pseudo-isomorphic to $\mathbb{Z}_p[S_\infty]^-$. □

Corollary 2.6. — *Let $\underset{\to\infty}{\widetilde{W}} = \varinjlim_m \widetilde{W}_m$ be the inductive limit of the $\widetilde{W}_m$'s. For all $n \geq n_d$ we have a (group)-isomorphism $H^1(\Gamma_n, \underset{\to\infty}{\widetilde{W}}) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{s^+}$ and $H^2(\Gamma_n, \underset{\to\infty}{\widetilde{W}})$ is trivial.*

Proof. — The groups $H^i(\Gamma_n, \underset{\to\infty}{\widetilde{W}})$ are isomorphic to the inductive limit with respect to the maps $H^i(G_{m,n}, \widetilde{W}_m) \longrightarrow H^i(G_{m+1,n}, \widetilde{W}_{m+1})$ induced by the couples of maps $G_{m+1,n} \to G_{m,n}$ and $\widetilde{W}_m \to \widetilde{W}_{m+1}$ (see proposition 1.5.1 of [**NSW**]). By proposition 2.4 the $\widetilde{W}_m$ satisfies Galois descent, therefore the inflation maps $H^1(G_{m,n}, \widetilde{W}_m) \longrightarrow$

$H^1(G_{m+1,n}, \widetilde{W}_{m+1})$ are injectives. As $H^1(G_{m,n}, \widetilde{W}_m) \simeq (\mathbb{Z}/p^{m-n})^{s^+}$ the first limit is $(\mathbb{Q}_p/\mathbb{Z}_p)^{s^+}$. Triviality of all H^2 's follows from that of all the $\widehat{H}^0$'s. □

We want to describe the $G_{m,n}$-cohomology of the $\widetilde{C}_m$'s and we will use a variation on the above method used for the $\widetilde{W}_m$'s. As at the beginning of the proof of proposition 2.4 we have an isomorphism $\mathrm{Ker}((\overline{C}_\infty)_{\Gamma_m} \longrightarrow \widetilde{C}_m) \cong (\overline{U}'_\infty/\overline{C}_\infty)^{\Gamma_m}$. But $\overline{U}'_\infty/\overline{C}_\infty$ may have a non trivial finite submodule. Indeed we have sequence

$$(1) \qquad 0 \longrightarrow KN_\infty^{\Gamma_m} \longrightarrow (\overline{U}'_\infty/\overline{C}_\infty)^{\Gamma_m} \longrightarrow (\overline{U}'_\infty/\overline{W}_\infty)^{\Gamma_m} \longrightarrow \text{finite},$$

extracted from the lemma 2.3 applied to the sequence $0 \to KN_\infty \to \overline{U}'_\infty/\overline{C}_\infty \to \overline{U}'_\infty/\overline{W}_\infty \to 0$.

Definition 2.7. — We define $V_m := (\overline{U}'_\infty/\overline{C}_\infty)^{\Gamma_m}/KN_\infty^{\Gamma_m}$.

From the sequence (1) and lemma 2.5 we see that V_m is $\mathbb{Z}_p$-free of rank s^+ and the natural action by Γ_{n_d} on V_m is trivial. Hence for $n \geq n_d$ the $G_{m,n}$-cohomology of V_m is just s^+ copies of the cohomology of $\mathbb{Z}$. We will get the cohomology of $\widetilde{C}_m$ by going through the cohomology of the sequence :

$$(2) \qquad 0 \longrightarrow V_m \longrightarrow (\overline{C}_\infty)_{\Gamma_m}/KN_\infty^{\Gamma_m} \longrightarrow \widetilde{C}_m \longrightarrow 0$$

Before this computation we will prove the existence of a first layer n from which our result will apply and try to make it as precise as possible.

Lemma 2.8. —

1. *There exists an n such that for all integers $m \geq n$ we have*
$$\left(\overline{U}'_\infty/\overline{C}_\infty\right)^{\Gamma_m} = \left(\overline{U}'_\infty/\overline{C}_\infty\right)^{\Gamma_n}.$$
2. *For any integer n satisfying 1, the group Γ_n acts trivially on KN_∞.*
3. *If n satisfies 1, then $n \geq n_d$.*
4. *If $KN_\infty = 0$ then the integer n_d of definition 2.2 satisfies 1.*

Proof. — The Λ-module $\overline{U}'_\infty/\overline{C}_\infty$ is finitely generated. The sequence of submodules $\left((\overline{U}'_\infty/\overline{C}_\infty)^{\Gamma_m}\right)_{m\in\mathbb{N}}$ is increasing. Part 1 of the lemma follows by noetherianity. Part 2 comes from the inclusion $KN_\infty \subset \overline{U}'_\infty/\overline{C}_\infty$. Suppose that some $x \in KN_\infty$ is such that $\sigma x \neq x$ for some $\sigma \in \Gamma_n$. But x, as an element of the finite module KN_∞, must be fixed by some Γ_m for $m > n$. Hence $x \in \left(\overline{U}'_\infty/\overline{C}_\infty\right)^{\Gamma_m} \setminus \left(\overline{U}'_\infty/\overline{C}_\infty\right)^{\Gamma_n}$ and n does not satisfy 1. This shows assertion 2. For assertion 3, and 4 we have seen in the proof of lemma 2.5 that $(\overline{U}'_\infty/\overline{W}_\infty)^{\Gamma_m}$ embeds with finite cokernel into $\mathbb{Z}_p[S_\infty]$ for all $m \geq n_d$. Now Γ_{n_d} acts trivially on $\mathbb{Z}_p[S_\infty]$ and if $n_d > 0$ then $\mathbb{Z}_p[S_\infty]^{\Gamma_{n_d-1}}$ is not of finite index in $\mathbb{Z}_p[S_\infty]$. This proves 3 and 4. □

Definition 2.9. — We will denote $n_{U'C}$ the smallest integer n such that for all integers $m \geq n$ the equality $\left(\overline{U}'_\infty/\overline{C}_\infty\right)^{\Gamma_m} = \left(\overline{U}'_\infty/\overline{C}_\infty\right)^{\Gamma_n}$ holds.

This number $n_{U'C}$ is the first nonexplicit part of our "asymptotic" approach. Other approaches found in the literature simply assume hypotheses that imply $KN_\infty = 0$ and in such case we simply have $n_{U'C} = n_d$.

Theorem 2.10. — *Let s^+ be as in proposition 2.4. Let n be greater than or equal to $n_{U'C}$. Let V_m be the $\mathbb{Z}_p$-free module of rank s^+ and trivial Γ_{n_d}-action of definition 2.7. For every integer i let $KN_\infty[p^i]$ be the kernel of multiplication by p^i in KN_∞. For every $m \geq n$ we have*

$$\widehat{H}^0(G_{m,n}, \widetilde{C}_m) \cong KN_\infty[p^{m-n}]$$

and a (group split) exact sequence

$$0 \longrightarrow KN_\infty/p^{m-n} \longrightarrow H^1(G_{m,n}, \widetilde{C}_m) \longrightarrow V_m/p^{m-n} \longrightarrow 0\ .$$

Proof. — Fix $m \geq n \geq n_{U'C}$. To describe the cohomology of the sequence (2), we first compute the cohomology of $(\overline{C}_\infty)_{\Gamma_m}/KN_\infty$. From the sequence $0 \to \overline{C}_\infty \to \overline{W}_\infty \to KN_\infty \to 0$ we obtain by lemma 2.3 a sequence

$$0 \longrightarrow (\overline{C}_\infty)_{\Gamma_m}/KN_\infty \longrightarrow (\overline{W}_\infty)_{\Gamma_m} \longrightarrow KN_\infty \longrightarrow 0.$$

As $(\overline{W}_\infty)_{\Gamma_m}$ is $G_{m,n}$-cohomologicaly trivial we see that $(\overline{C}_\infty)_{\Gamma_m}/KN_\infty$ has the $G_{m,n}$-cohomology of KN_∞ shifted once. The exact hexagone of cyclic Tate cohomology associated to the sequence (2) is now :

$$\begin{array}{ccccc}
V_m/p^{m-n} & \xrightarrow{\ \alpha\ } & KN_\infty[p^{m-n}] & \longrightarrow & \widehat{H}^0(G_{m,n}, \widetilde{C}_m) \\
\uparrow & & & & \downarrow \\
\widehat{H}^{-1}(G_{m,n}, \widetilde{C}_m) & \longleftarrow & KN_\infty/p^{m-n} & \longleftarrow & 0
\end{array}$$

To conclude the proof of the theorem it remains now to prove that the map α of this diagram is the 0 map. This map α is induced by the map $\delta\colon (\overline{U}'_\infty/\overline{C}_\infty)^{\Gamma_m} \longrightarrow (\overline{C}_\infty)_{\Gamma_m}$ of the lemma 2.3 applied to the sequence $0 \to \overline{C}_\infty \to \overline{U}'_\infty \to \overline{U}'_\infty/\overline{C}_\infty \to 0$. To define this map δ let us pick a choice of a generator γ_m of Γ_m, and for later use let us denote γ_n the generator of Γ_n such that $\gamma_n^{p^{m-n}} = \gamma_m$. Then the map δ sends any coset $\overline{u} \in \overline{U}'_\infty/\overline{C}_\infty$ such that $(\gamma_m - 1)u \in \overline{C}_\infty$ to the coset of $(\gamma_m - 1)u$ in $(\overline{C}_\infty)_{\Gamma_m}$. Note that $\gamma_m - 1 = \nu_{m,n}(\gamma_n - 1)$, where $\nu_{m,n} = \sum_{i=0}^{p^{m-n}-1} \gamma_n^i \in \Lambda$ acts like the algebraic norm of the extension F_m/F_n on any Γ_m-trivial module. But the integers $m \geq n$ have been chosen in such a way that $(\overline{U}'_\infty/\overline{C}_\infty)^{\Gamma_m} = (\overline{U}'_\infty/\overline{C}_\infty)^{\Gamma_n}$ and therefore we have $(\gamma_m - 1)u = \nu_{m,n}(\gamma_n - 1)u$, where $(\gamma_n - 1)u \in \overline{C}_\infty$. As by definition we have

$$\widehat{H}^0(G_{m,n}, (\overline{C}_\infty)_{\Gamma_m}/KN_\infty) = \frac{((\overline{C}_\infty)_{\Gamma_m}/KN_\infty)^{G_{m,n}}}{\nu_{m,n}((\overline{C}_\infty)_{\Gamma_m}/KN_\infty)},$$

this shows that α is indeed the 0 map and concludes the proof of theorem 2.10. □

Corollary 2.11. — *Let $\underset{\to\infty}{\widetilde{C}} = \varinjlim_m \widetilde{C}_m$ be the inductive limit of the $\widetilde{C}_m$'s. For any $n \geq n_{U'C}$ we have a (group)-isomorphism $H^1(\Gamma_n, \underset{\to\infty}{\widetilde{C}}) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{s^+}$ and $H^2(\Gamma_n, \underset{\to\infty}{\widetilde{C}})$ is trivial.*

Proof. — As Γ_n is pro-p-free, $H^2(\Gamma_n, \underset{\to\infty}{\widetilde{C}})$ is divisible; and its order is bounded by the finite order of KN_∞, hence $H^2(\Gamma_n, \underset{\to\infty}{\widetilde{C}})$ is trivial. Let us compute the $H^1(\Gamma_n, \underset{\to\infty}{\widetilde{C}})$. Extracted mutatis-mutandis from the proof of the theorem 2.10 we have the exact sequence for all $m >> 0$:

$$0 \longrightarrow H^1(G_{m,n}, (\overline{C}_\infty)_{\Gamma_m}/KN_\infty) \longrightarrow H^1(G_{m,n}, \widetilde{C}_m) \longrightarrow H^2(G_{m,n}, V_m) \longrightarrow 0.$$

Of course we have for $m >> 0$ the isomorphisms $H^1(G_{m,n}, (\overline{C}_\infty)_{\Gamma_m}/KN_\infty) \simeq KN_\infty$ and $H^2(G_{m,n}, V_m) \simeq V_m/p^{m-n}$ and we recover the exact sequence :

$$0 \longrightarrow KN_\infty \longrightarrow H^1(G_{m,n}, \widetilde{C}_m) \longrightarrow V_m/p^{m-n} \longrightarrow 0.$$

We want to take inductive limits on this sequence, and for that we need to clarify what are the extension maps going up from the mth to the $(m+1)$th step for the modules KN_∞ and the modules V_m/p^{m-n}. The finite module of universal norms KN_∞ stabilizes with respect to natural (going down) norm maps and composition with extension maps gives multiplication by p. Therefore the inductive limit with respect to extension maps of KN_∞ is trivial. On the contrary the modules $(V_m)_m$ stabilize with respect to extension maps already from the first step to V_n for $m \geq n \geq n_{U'C}$. So for fixed $n \geq n_{U'C}$ the inductive limit $\varinjlim V_m/p^{m-n}$ is $V_n \otimes \mathbb{Q}_p/\mathbb{Z}_p \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{s^+}$. □

3. Cohomology of circular units

We will try and recover the cohomology of $\overline{C}_m$ and $\overline{W}_m$ from that of their universal norms $\widetilde{C}_m$ and $\widetilde{W}_m$. For $\overline{C}_m$ the method is not new and has already been used in [**N**] or [**NL**] and originally in [**B1**]. Here we only encounter another slight difficulty coming from the non triviality of $\widehat{H}^0(G_{m,n}, \widetilde{C}_m)$. For $\overline{W}_m$ our result is original.

Lemma 3.1. — *For all n let I_n be the inertia subfield for p in $F_n/\mathbb{Q}$. We have $I_{n+1} = I_n$ as soon as F_{n+1}/F_n ramifies. Let $I = \bigcup_n I_n$ be the inertia subfield for p in $F_\infty/\mathbb{Q}$.*

1. *For all n we have equality*

$$\overline{C}_n = \widetilde{C}_n \overline{C}(I_n),$$

hence for all n such that F_{n+1}/F_n ramifies we have equality

$$\overline{C}_n = \widetilde{C}_n \overline{C}(I).$$

2. *There exist an integer n such that for all integers $m \geq n$ we have*

$$\widetilde{C}_m \bigcap C(I) = \widetilde{C}_n \bigcap C(I).$$

Proof. — Assertion 1 is lemme 2.5 of [**B2**]. The $\mathbb{Z}_p$-module $C(I)$ is finitely generated. The sequence of submodules $\left(\widetilde{C}_m \bigcap C(I)\right)_{m\in\mathbb{N}}$ is increasing. Assertion 2 follows by noetherianity. □

Definition 3.2. — We will denote n_C the smallest non negative integer such that F_{n_C+1}/F_n ramifies and such that property 2 of lemma 3.1 holds.

This number n_C is our second nonexplicit asymptotic constant. However if we assume $KN_\infty = 0$ then theorem 2.10 shows that the $\widetilde{C}_m$'s satisfy Galois descent and that proves the equality $n_C = n_i$, where n_i is the smallest non negative integer such that F_{n_i+1}/F_{n_i} ramifies.

After KN_∞ there is another module that we have to consider as a parameter depending on F. This module is essentially the one denoted Φ in [**N**] and that we will define in our more general context now :

Definition 3.3. — We call universal co-norms of circular units and denote by Φ_m the quotient module

$$\Phi_m = \overline{C}_m/\widetilde{C}_m.$$

We state and prove now the properties of Φ_m which are needed to compute the cohomology of $\overline{C}_m$. In particular we fix its asymptotic behavior, so that we may consider it as a parameter (even if it is a non explicit and asymptotic one !). We will give somewhat more explicit informations about this module and examine some examples in section 4.

Lemma 3.4. — *Recall that n_i is the first nonnegative integer such that F_{n_i+1}/F_{n_i} ramifies.*

1. *For all m the $\mathbb{Z}_p$-rank of Φ_m is $s_m^+ - 1$ where s_m^+ is the cardinal of the set S_m^+ of places dividing p in the maximal real subfield F_m^+ of F_m.*
2. *For all m the group Γ_{n_i} acts trivially on Φ_m.*
3. *For any $m \geq n_C$ the extension map $\overline{C}_{n_C} \longrightarrow \overline{C}_m$ induces an isomorphism*

$$\Phi_{n_C} \cong \Phi_m.$$

Proof. — This lemma is an easy generalization of lemma-definition 2.2 in [**NL**], but for the convenience of the reader we reprove it. To compute the rank of Φ_m just use the exact sequence

$$0 \longrightarrow (\overline{U}'_\infty/\overline{C}_\infty)^{\Gamma_m} \longrightarrow (\overline{C}_\infty)_{\Gamma_m} \longrightarrow \overline{C}_m \longrightarrow \Phi_m \longrightarrow 0\ .$$

Let $r = [F^+ : \mathbb{Q}]$. Then the Λ-module $\overline{C}_\infty$ has trivial Γ-invariants and rank r. It follows that $(\overline{C}_\infty)_{\Gamma_m}$ has $\mathbb{Z}_p$-rank equal to rp^m. Also the $\mathbb{Z}_p$-module $\overline{C}_m$ is of finite index in $\overline{U}_m$ and has rank $rp^m - 1$ by Dirichlet's theorem. The module $(\overline{U}'_\infty/\overline{C}_\infty)^{\Gamma_m}$ has the same rank as $(\overline{U}'_\infty/\overline{W}_\infty)^{\Gamma_m}$ i.e. s_m^+ by assertion 2 of lemma 2.5. This proves 1. Assertions 2 and 3 follow from lemma 3.1. Indeed for all m we have

$$\Phi_m = \overline{C}_m/\widetilde{C}_m = \widetilde{C}_m C(I_m)/\widetilde{C}_m \cong C(I_m)/\widetilde{C}_m \bigcap C(I_m).$$

For all m the action of Γ_{n_i} is trivial on I_m, which proves 2 and for all $m \geq n_C$ the extension map $\Phi_{n_C} \longrightarrow \Phi_m$ commutes with the identity map in

$$\overline{C}(I)/(\widetilde{C}_{n_C} \cap \overline{C}(I)) = \overline{C}(I)/(\widetilde{C}_m \cap \overline{C}(I)),$$

which proves 3. $\square$

In the sequel we will abbreviate Φ_{n_C} to simply Φ. Recall that from theorem 2.10 we have for all $m \geq n \geq n_{U'C}$ isomorphisms $\widehat{H}^0(G_{m,n}, \widetilde{C}_m) \cong KN_\infty[p^{m-n}]$ and exact sequences :

$$0 \longrightarrow KN_\infty/p^{m-n}KN_\infty \longrightarrow \widehat{H}^{-1}(G_{m,n}, \widetilde{C}_m) \longrightarrow (\mathbb{Z}/p^{m-n})^{s^+} \longrightarrow 0 .$$

Theorem 3.5. — *Let n be any integer not less than both $n_{U'C}$ and n_C. Then for all $m \geq n$ we have exact sequences*

$$0 \longrightarrow KN_\infty[p^{m-n}] \longrightarrow \widehat{H}^0(G_{m,n}, \overline{C}_m) \longrightarrow \Phi/p^{m-n}\Phi \longrightarrow 0$$

and

$$0 \longrightarrow H^1(G_{m,n}, \widetilde{C}_m) \longrightarrow H^1(G_{m,n}, \overline{C}_m) \longrightarrow \Phi[p^{m-n}] \longrightarrow 0 .$$

Proof. — Fix $m \geq n$ with $n \geq \max(n_C, n_{U'C})$. As $G_{m,n}$ is cyclic we may and will compute $\widehat{H}^{-1}$ during the proof. This proof consists in splitting into two pieces the exact hexagone of cyclic Tate cohomology groups associated to the exact sequence $0 \longrightarrow \widetilde{C}_m \longrightarrow \overline{C}_m \longrightarrow \Phi \longrightarrow 0$. At first sight this exact hexagone is

$$\begin{array}{ccccc}
KN_\infty[p^{m-n}] & \xrightarrow{\varphi_0} & \widehat{H}^0(G_{m,n}, \overline{C}_m) & \longrightarrow & \Phi/p^{m-n}\Phi \\
\delta_{-1}\big\uparrow & & & & \big\downarrow\delta_0 \\
\Phi[p^{m-n}] & \longleftarrow & \widehat{H}^{-1}(G_{m,n}, \overline{C}_m) & \underset{\varphi_{-1}}{\longleftarrow} & \widehat{H}^{-1}(G_{m,n}, \widetilde{C}_m).
\end{array}$$

So we only have to prove that the two δ maps are both the 0 map or equivalently that the φ maps are injective. The injectivity of φ_{-1} follows from the stabilization with respect to extension maps of the quotients Φ_m. Indeed from this stabilization property we can write $\overline{C}_m = \overline{C}_n\widetilde{C}_m$ and therefore for all generator σ of $G_{m,n}$ we get $(\sigma - 1)(\overline{C}_m) = (\sigma - 1)(\widetilde{C}_m)$ which proves that φ_{-1} is injective.

The kernel of φ_0 is by definition the quotient $N_{m,n}(\overline{C}_m) \cap \widetilde{C}_m/N_{m,n}(\widetilde{C}_m)$, where $N_{m,n}$ stands for the norm map of F_m/F_n. But by assertion 2 of lemma 3.1 we have

$$N_{m,n}(\overline{C}_m) = N_{m,n}(\widetilde{C}_m\overline{C}(I)) = N_{m,n}(\widetilde{C}_m)\overline{C}(I)^{p^{m-n}}.$$

Hence the equalities

$$\operatorname{Ker}\varphi_0 = \frac{\left(N_{m,n}(\widetilde{C}_m)\overline{C}(I)^{p^{m-n}}\right)\bigcap \widetilde{C}_m}{N_{m,n}(\widetilde{C}_m)} = \frac{N_{m,n}(\widetilde{C}_m)\left(\overline{C}(I)^{p^{m-n}}\bigcap \widetilde{C}_m\right)}{N_{m,n}(\widetilde{C}_m)}.$$

But by the definition of n_C for $m \geq n \geq n_C$ we have $\overline{C}(I) \cap \widetilde{C}_m = \overline{C}(I) \cap \widetilde{C}_n$ and by definition of $\widetilde{C}_n$ we have $\widetilde{C}_n \subset N_{m,n}(\widetilde{C}_m)$. This proves that $\operatorname{Ker}\varphi_0 = 0$. □

Corollary 3.6. — *Let $\overrightarrow{C}_\infty$ be the inductive limit with respect to extension maps of the $\overline{C}_m$'s. Let* $\operatorname{Tor}(\Phi)$ *be the $\mathbb{Z}_p$-torsion of Φ. For all n not less than both n_C and $n_{U'C}$ we have (group) isomorphisms :*

$$H^1(\Gamma_n, \overrightarrow{C}_\infty) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{s^+} \oplus \operatorname{Tor}(\Phi),$$

and

$$H^2(\Gamma_n, \overrightarrow{C}_\infty) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{s^+-1}.$$

Proof. — As before, when taking inductive limits, all contributions from the module KN_∞ vanish. The module Φ, contrary to KN_∞, stabilizes with respect to extension maps. Therefore extensions for $\Phi[p^k] \longrightarrow \Phi[p^{k+1}]$ ultimately are isomorphisms and give $\mathrm{Tor}(\Phi)$ as inductive limit, while extensions for $\Phi/p^k \longrightarrow \Phi/p^{k+1}$ are right from the start induced by multiplication by p and give $\Phi \otimes (\mathbb{Q}_p/\mathbb{Z}_p) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{s^+-1}$ as inductive limit. The sequence with H^1 is group split because $(\mathbb{Q}_p/\mathbb{Z}_p)^{s^+}$ is an injective group. □

Finally we want to describe the cohomology of the W_m's. We use the same strategy as for the C_m's and for that, we first have to prove the stabilization by extension maps of the $\overline{W}_m/\widetilde{W}_m$'s. We will deduce this stabilization from the one of the Φ_m's, which unfortunately gives us a even worst lower bound n_W from which our results apply.

Lemma 3.7. — *There Exists an n such that for any $m \geq n$ the extension map $\overline{W}_n \longrightarrow \overline{W}_m$ induces an isomorphism $\overline{W}_n/\widetilde{W}_n \cong \overline{W}_m/\widetilde{W}_m$.*

Proof. — First take an $n \geq n_C$, and consider the (tautological) diagram with exact rows :

$$\begin{array}{ccccccccc}
0 \to (\widetilde{W}_n \cap \overline{C}_n)/\widetilde{C}_n & \to & \overline{C}_n/\widetilde{C}_n & & \longrightarrow & & \overline{W}_n/\widetilde{W}_n & \to & \overline{W}_n/(\overline{C}_n\widetilde{W}_n) \to 0 \\
 & & & \searrow & & \nearrow & & & \\
(\dagger) & & & & \overline{C}_n/(\widetilde{W}_n \cap \overline{C}_n) & & & &
\end{array}$$

As $n \geq n_C$ the extension map $\Phi_n \longrightarrow \Phi_m$ is surjective and so is the extension map $\overline{C}_n/(\widetilde{W}_n \cap \overline{C}_n) \longrightarrow \overline{C}_m/(\widetilde{W}_n \cap \overline{C}_m)$. Recall that we have $n \geq n_C \geq n_i \geq n_d$, so that theorem 2.4 applies. In particular the $\widetilde{W}_m$'s satisfy Galois descent for $m \geq n$ and the extension map $\overline{W}_n/\widetilde{W}_n \longrightarrow \overline{W}_m/\widetilde{W}_m$ is injective. Now use the snake lemma on the right hand part of the diagram $(\dagger)$ with extension maps. This gives :

$$\begin{array}{ccccccccc}
 & & & & 0 & \longrightarrow & K_{n,m} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{C}_n/(\widetilde{W}_n \cap \overline{C}_n) & \longrightarrow & \overline{W}_n/\widetilde{W}_n & \longrightarrow & \overline{W}_n/(\overline{C}_n\widetilde{W}_n) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{C}_m/(\widetilde{W}_m \cap \overline{C}_m) & \longrightarrow & \overline{W}_m/\widetilde{W}_m & \longrightarrow & \overline{W}_m/(\overline{C}_m\widetilde{W}_m) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & CoK_{n,m} & \longrightarrow & CoK'_{n,m} & \longrightarrow & 0.
\end{array}$$

It follows that the kernels $K_{n,m}$ are trivial and that the co-kernels $CoK_{n,m}$ and $CoK'_{n,m}$ are isomorphic. By the triviality of the kernels we see that the orders of the finite groups $\overline{W}_n/\overline{C}_n\widetilde{W}_n$ are increasing with n, but must stabilize because they are uniformly bounded by the maximal of the orders of $\overline{C}_n/\overline{W}_n$ (which exists according to main result of [**KN**]). So for an n greater than n_C and such that the order of $\overline{W}_n/\overline{C}_n\widetilde{W}_n$ is maximal, we get the triviality of the two cokernels $CoK_{n,m}$. □

Definition 3.8. — We will denote by n_W the smallest integer not less than n_d and such that for all $m \geq n$ the extension map $\overline{W}_n/\widetilde{W}_n \longrightarrow \overline{W}_m/\widetilde{W}_m$ is an isomorphism. We will abbreviate $\overline{W}_m/\widetilde{W}_m$ to ΦW_m and $\overline{W}_{n_W}/\widetilde{W}_{n_W}$ to simply ΦW.

Using diagram (†) again we see that ΦW_m and Φ_m have the same $\mathbb{Z}_p$-ranks, namely $s_m^+ - 1$, and of course that the natural action of Γ_{n_W} on all ΦW_m's is trivial.

Theorem 3.9. — *For all $m \geq n \geq n_W$ we have isomorphisms :*

$$\widehat{H}^0(G_{m,n}, \overline{W}_m) \cong \Phi W/p^{m-n}$$

and (group split) exact sequences :

$$0 \longrightarrow (\mathbb{Z}/p^{m-n})^{s^+} \longrightarrow H^1(G_{m,n}, \overline{W}_m) \longrightarrow \Phi W[p^{m-n}] \longrightarrow 0$$

Proof. — Using lemma 3.7 the proof of this theorem is the same as the proof of theorem 3.5. It is even actually easier because when splitting in two pieces the exact hexagone associated to the sequence $0 \longrightarrow \widetilde{W}_m \longrightarrow \overline{W}_m \longrightarrow \Phi W \longrightarrow 0$, we only have to prove the injectivity of the map corresponding to φ_{-1} by virtue of the triviality of $\widehat{H}^0(G_{m,n}, \widetilde{W}_m)$. This injectivity in turn follows from the stabilization of ΦW_m with respect to extension maps. □

Corollary 3.10. — *Let $\vec{W}_\infty$ be the inductive limit with respect to extension maps of the $\overline{W}_m$. Let $\mathrm{Tor}(\Phi W)$ be the $\mathbb{Z}_p$-torsion of ΦW. For all n greater than or equal to n_W we have (group) isomorphisms :*

$$H^1(\Gamma_n, \vec{W}_\infty) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{s^+} \oplus \mathrm{Tor}(\Phi W),$$

and

$$H^2(\Gamma_n, \vec{W}_\infty) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{s^+-1}.$$

Proof. — The proof is exactly the same as the one of corollary 3.6. □

4. Universal co-norms of circular units

In this section we want to investigate the module $\Phi_m = \overline{C}_m/\widetilde{C}_m$ which, together with KN_∞, describes the cohomology of the $\overline{C}_m$'s. By lemma 3.4 we already know that its $\mathbb{Z}_p$-rank is $s_m^+ - 1$ where s_m^+ is the number of places of F_m^+ dividing p. As we have $\Phi_m = \overline{C}_m/\widetilde{C}_m = \overline{C}_m^+/\widetilde{C}_m^+$ we may suppose without loss of generality that F is totally real. Next, as the rank of Φ_m is known, we should concentrate on its $\mathbb{Z}_p$-torsion, let's say $\mathrm{Tor}(\Phi_m)$. To give examples of non trivial $\mathrm{Tor}(\Phi_m)$ we state and prove a lemma which provides also criteria for triviality of $\mathrm{Tor}(\Phi_m)$.

Lemma 4.1. — *Recall that I_m is the maximal subfield of F_m such that p does not ramify in $I_m/\mathbb{Q}$, and that $\overline{\mathrm{Cyc}}(I_m)$ is the p-completion of its module of cyclotomic numbers. Let σ_p be the Fröbenius automorphism of I_m.*

1. *We have an exact sequence of finite groups*

$$0 \longrightarrow \widetilde{C}_m \cap \overline{C}(I_m)/\overline{\mathrm{Cyc}}(I_m)^{\sigma_p - 1} \longrightarrow \widehat{H}^{-1}(\langle\sigma_p\rangle, \overline{\mathrm{Cyc}}(I_m)) \longrightarrow \mathrm{Tor}(\Phi_m) \longrightarrow 0.$$

2. *Assume that $KN_\infty = 0$ and that p does not ramify in F, then we have an isomorphism* $\mathrm{Tor}(\Phi_m) \cong \widehat{H}^{-1}(\langle\sigma_p\rangle, \overline{\mathrm{Cyc}}(I_m))$

Proof. — By lemma 3.1 we have an isomorphism $\Phi_m \cong C(I_m)/C(I_m) \cap \widetilde{C}_m$. Using distribution relations it is easy to check that $\overline{\mathrm{Cyc}}(I_m)^{\sigma_p - 1} \subset \overline{C}(I_m) \cap \widetilde{C}_m$. By mere rank computation the quotient $C(I_m) \cap \widetilde{C}_m/\mathrm{Cyc}(I_m)^{\sigma_p-1}$ is finite. Let D_m be the maximal subfield of F_m such that p (totally) splits in $D_m/\mathbb{Q}$, and let N_{I_m/D_m} be the norm map. Write $\overline{\mathrm{Cyc}}(I_m)[N_{I_m/D_m}]$ for the kernel of this norm on $\overline{\mathrm{Cyc}}(I_m)$. Note that $\overline{\mathrm{Cyc}}(I_m)[N_{I_m/D_m}] \subset \overline{C}(I_m)$. Then $\overline{\mathrm{Cyc}}(I_m)[N_{I_m/D_m}]$ contains $\mathrm{Cyc}(I_m)^{\sigma_p-1}$ with finite index and is maximal with respect to that property inside $\overline{C}(I_m)$ because $N_{I_m/D_M}(\overline{\mathrm{Cyc}}(I_m)) \subset \overline{\mathrm{Cyc}}(D_m)$ is torsion-free. To summarize, we have inclusions of modules with finite index :

$$\overline{\mathrm{Cyc}}(I_m)^{\sigma_p-1} \subset \widetilde{C}_m \cap \overline{C}(I_m) \subset \overline{\mathrm{Cyc}}(I_m)[N_{I_m/D_m}]. \tag{3}$$

It follows that $\mathrm{Tor}(\Phi_m) = \overline{\mathrm{Cyc}}(I_m)[N_{I_m/D_m}]/\overline{C}(I_m) \cap \widetilde{C}_m$ and the exact sequence in 1 becomes tautological. Moreover, if we assume that p does not ramify in F and $KN_\infty = 0$, then we get $\widetilde{C}_m \cap \overline{C}(I_m) = \widetilde{C}_m \cap \overline{C}(F)$ and $n_{U'C} = 0$. By theorem 2.10 the sequence of modules $(\widetilde{C}_m)_m$ satisfies Galois descent and therefore $\widetilde{C}_m \cap \overline{C}(I_m) = \widetilde{C}_0$. Now it is not difficult to see on a system of generators of $\overline{\mathrm{Cyc}}(I_m)$ that, in that case, distribution relations imply the equality $\widetilde{C}_0 = \overline{\mathrm{Cyc}}(I_m)^{\sigma_p-1}$, so that 2 follows from 1. □

Using the exact sequence of 1, we see that $\mathrm{Tor}(\Phi_m)$ turns out to be trivial as soon as p does not divide the order of σ_p. Intuitively, in the sequence of inclusions (3), any couple of powers of p should occur in some cases as a couple of indices. We prove something far less ambitious which provides the simplest example of non-trivial $\mathrm{Tor}(\Phi_m)$.

Proposition 4.2. — *Let F be a real abelian field of degree p such that σ_p generates $\mathrm{Gal}(F/\mathbb{Q})$. Then we have $KN_\infty = 0$ and, for all m, isomorphisms $\Phi_0 \xrightarrow{\sim} \Phi_m$. Moreover Φ_0 is finite of order p if and only if at least two distinct rational primes ramify in F and is trivial if and only if a single prime $\ell \neq p$ ramifies in F.*

Proof. — In the present case we have $s_m^+ = 1$ for all m, so that $\Phi_m = \mathrm{Tor}(\Phi_m)$. Then $\mathrm{Gal}(F/\mathbb{Q})$ is cyclic and p is unramified in F so that KN_∞ is trivial (see [**B1**]). By 2 of lemma 4.1 we deduce $\Phi_m = \Phi_0 = \widehat{H}^{-1}(\langle\sigma_p\rangle, \overline{\mathrm{Cyc}}(F))$. In this case the module $\mathrm{Cyc}(F)$ is generated by the single number $N_{\mathbb{Q}(\zeta_f)/F}(1-\zeta_f)$ where f is the conductor of F. If f is a single prime power then this module is Galois free and has trivial cohomology. Else this module is isomorphic to $\mathbb{Z}[\zeta_p]$, with residue field $\mathbb{F}_p$ as $H^{-1}(\mathbb{Z}/p, \mathbb{Z}[\zeta_p])$. □

To conclude this section of examples, let us be completely explicit. Take $p = 3$, $\ell_1 = 7$ and $\ell_2 = 13$. Observe that 3 is not a third power modulo 7 neither modulo 13. Then for $i = 1$ or 2 the field $\mathbb{Q}(\zeta_{\ell_i})$ contains a cubic subfield (say F_i) which admits $\mathbb{F}_{27}$ as residue field (in other word the ideal (3) remains prime in F_i). Let L be the compositum F_1F_2. Then $\mathrm{Gal}(L/\mathbb{Q})$ is of type $[3,3]$ and therefore L admits 4 cubic subfields. These subfields are the inertia subfield at ℓ_2, which is F_1, the inertia subfield at ℓ_1, which is F_2, the decomposition subfield at 3, say D, and a fourth subfield F which has conductor $f = \ell_1\ell_2$ and in which the ideal (3) remains prime. In these very simple cases, using proposition 4.2, we have trivial Φ_0 for the fields F_i, an example of finite Φ_0 of order p for the field F and an example of $\mathbb{Z}_p$-free Φ_0 of rank 2 for the field D.

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