On modified circular units and annihilation of real classes

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January 24, 2005

Abstract

For an abelian totally real number field F and an odd prime number p which splits totally in F, we present a functorial approach to special "p-units" previously built by D. Solomon using "wild" Euler systems. This allows us to prove a conjecture of Solomon on the annihilation of the p-class group of F (in the particular context here), as well as related annihilation results and index formulae.

0 Introduction

Let F be an imaginary abelian field, $G = \mathrm{Gal}(F/\mathbb{Q})$, and let $\mathrm{Cl}_F^{\pm}$ be the "plus" and "minus" parts defined by complex conjugation acting on the class group Cl_F of F. The classical Stickelberger theorem asserts that the Stickelberger ideal, say $\mathrm{Stick}(F) \subset \mathbb{Z}[G]$, is an annihilator of Cl_F^-. Sinnott has shown that the index $(\mathbb{Z}[G]^- : \mathrm{Stick}(F)^-)$ is essentially "equal" to the order h_F^- of Cl_F^-. Using the p-adic point of view, these results fit into the theory of Iwasawa and of p-adic L-functions via what is still called the Main Conjecture, even though it is now a theorem (Mazur-Wiles).

For Cl_F^+, our knowledge is more fragmented. For simplicity take $p \neq 2$. The Main Conjecture relates (in a "numerical" way) the p-part X_F^+ of Cl_F^+ to the p-part of U_{F^+}/C_{F^+}, where U_{F^+} (resp. C_{F^+}) is the group of units (resp. circular units) of the maximal real subfield F^+ of F. In the semi-simple case, it also gives an ideal $MW(F)$ which annihilates a certain part of X_F^+ (the maximal subgroup acting trivially on the intersection of the $\mathbb{Z}_p$-cyclotomic extension of F^+ with the p-Hilbert class field of F^+). In the papers [So1], [So2] Solomon constructed what could be considered as real analogues of Gauß sums and of Stickelberger's element, from which he conjectured an annihilation of X_F^+. Before sketching Solomon's construction let us recall some main principles of the demonstration of Stickelberger's theorem using Gauß sums : for all prime ideals $\wp$ not dividing the conductor of F, one constructs the Gauß sum $g(\wp)$ (which belongs to some cyclotomic extension of F), and if x is a denominator of the

[0]MSC number primary : 11R23, 11R18, 11R20

Stickelberger element $\theta_F \in \mathbb{Q}[G]$, then $(g(\wp)^x) = \wp^{x\theta_F}$. We stress two important ingredients :

- one can always choose in the class of $\wp$ prime ideals $\mathfrak{L}$ which split in $F/\mathbb{Q}$.
- the Gauß sum is an ℓ-unit ($\mathfrak{L} \mid \ell$), and the computation of its $\mathfrak{L}$-adic valuation plays a crucial role.

We return to Solomon's construction. For simplicity, we now assume that F is a (totally) real field, and p an odd prime number splitting in F. From a fixed norm coherent sequence of cyclotomic numbers (in the extension $F(\mu_{p^\infty})$), Solomon constructs, using a process which could be seen as a "wild" variant of Kolyvagin-Rubin-Thaine's method, a special "(p)-unit" $\kappa(F,\gamma)$ (which depends on the choice of a topological generator of $\mathrm{Gal}(F(\mu_{p^\infty})/F(\mu_p))$, γ say). In fact $\kappa(F,\gamma) \in U'_F \otimes \mathbb{Z}_p$ where U'_F is the group of (p)-units of F. The main result of [So1] describes the $\mathfrak{P}$-adic valuation of $\kappa(F,\gamma)$. Let f be the conductor of F, put $\zeta_f = \exp(2i\pi/f)$, and define $\varepsilon_F = N_{\mathbb{Q}(\zeta_f)/F}(1-\zeta_f)$. Then for $\mathfrak{P} \mid p$ we have the p-adic equivalence :

$$v_{\mathfrak{P}}(\kappa(F,\gamma)) \sim \frac{1}{p}\log_p(\imath_{\mathfrak{P}}(\varepsilon_F))$$

where $\imath_{\mathfrak{P}}$ is the embedding $F \hookrightarrow \mathbb{Q}_p$ defined by $\mathfrak{P}$. From this computation and fixing an embedding $\imath \colon F \hookrightarrow \mathbb{Q}_p$, Solomon introduces the element

$$\mathrm{sol}_F := \frac{1}{p}\sum_{g\in G}\log_p(\imath(\varepsilon_F^g))g^{-1} \in \mathbb{Z}_p[G],$$

and conjectures that this element annihilates the p-part X_F of the class group of F ([So1], conjecture 4.1; actually Solomon states a more general conjecture assuming only that p is unramified in F). Here some comments are in order :

- Solomon's construction is not a functorial one, which means that he is dealing with elements instead of morphisms (this objection may be addressed to most "Euler system" constructions); this may explain the abundance of technical computations required to show the main result or some (very) special cases of the conjecture.
- In the semi-simple case (i.e. $p \nmid [F:\mathbb{Q}]$), Solomon's conjecture is true but it gives nothing new since one can easily see that $\mathrm{sol}_F \in MW(F)$ in that case.

The object of this paper is to present Solomon's construction in a more functorial way, making a more wholehearted use of Iwasawa theory. This functorial approach is not gratuitous, since on the one hand it is an alternative to the technical computations in [So1],[So2], and on the other hand it allows us to prove a slightly modified version of Solomon's conjecture and other related annihilation results in the most important special case, namely when p splits in F (see theorem 5.4).

We now give an idea of our functorial construction and of its main derivatives :
Assuming the above mentioned hypotheses (F is totally real and p splits in F), following U_F and C_F, we introduce U'_F, the group of (p)-units of F, together with the p-adic completions $\overline{U}_F$, $\overline{C}_F$ and $\overline{U}'_F$. The natural idea would be to apply Iwasawa co-descent to the corresponding Λ-modules $\overline{U}_\infty$, $\overline{C}_\infty$, $\overline{U}'_\infty$, obtained by taking inverse limits up the cyclotomic $\mathbb{Z}_p$-extension of F, but since we are assuming that p splits in F, some natural homomorphisms, for example $(\overline{C}_\infty)_\Gamma \longrightarrow \overline{C}_F$, are trivial! The main point will be to construct other natural (but non trivial) morphisms, which will allow us to compare the modules at the level of F with the corresponding modules arising from Iwasawa co-descent. To fix the ideas let us see how to compare $\overline{C}_F$ with $(\overline{C}_\infty)_\Gamma$. We construct successively two homomorphisms :
1) The first one, whose construction essentially uses an analog of "Hilbert's theorem 90 in Iwasawa theory", allows us to map $(\overline{C}_\infty)_\Gamma$ to $\overline{U}'_F$ after "dividing by $(\gamma-1)$" (see theorem 2.7). One may even go to the quotient by $\overline{U}_F$, the natural projection being injective when restricted to the image of $(\overline{C}_\infty)_\Gamma$. So we have canonical maps :

$(\overline{C}_\infty)_\Gamma \xrightarrow{\alpha} \overline{U}'_F$ and $(\overline{C}_\infty)_\Gamma \xrightarrow{\beta} \overline{U}'_F/\overline{U}_F$. Then the image under β of an ad hoc element in $\overline{C}_\infty$ is nothing else but Solomon's element $\kappa(F,\gamma)$ and functorial computations with β allow us to solve all index questions arising in [So1],[So2].
2) The second homomorphism constructs a "bridge" linking Sinnott's exact sequence with the class-field exact sequence. More precisely, $\widetilde{C}''_F := \alpha((\overline{C}_\infty)_\Gamma)$ is considered as a submodule of $\overline{U}'_F/\overline{U}_F$ as in 1), while $\overline{C}_F$ is considered as a submodule of $\overline{U}'_F/\widehat{U}'_F$, where $\widehat{U}'_F$ is some kernel consisting of local cyclotomic norms, usually called the Sinnott-Gross kernel. A priori there doesn't exist any natural map from $\overline{U}'_F/\widehat{U}'_F$ to $\overline{U}'_F/\overline{U}_F$, since $\overline{U}_F \bigcap \widehat{U}'_F = \{1\}$, but when restricting to $\overline{C}_F$ and using the map β, one is able to construct a map $\varphi\colon \overline{C}_F \longrightarrow {}_{\mathrm{Tr}}(\widetilde{C}''_F)$ (the kernel of the algebraic trace of $\mathbb{Z}_p[G]$ acting on $\widetilde{C}''_F$), which is injective and whose cokernel may be computed (see theorem 4.5) .

Once the morphisms α, β, φ have been constructed one is able to compute the Fitting ideals in $\mathbb{Z}_p[G]$ of various codescent modules in terms of Solomon's elements. This yields annihilation results for various classes, e.g. ideal classes, but also quotients of global units or semi-local units modulo circular units (see theorem 5.7). Note that in the non semi-simple case, these differ from the ones obtained by means of usual (tame) Euler systems.

In recent developments ([BB2], [BG], [RW2]) around the "lifted root number conjecture" and its "twisted" versions (or "equivariant Tamagawa number" conjecture for Tate motives over $\mathbb{Q}$ in the language of Burns and Flach), Solomon's elements appear to play a role. In some special cases, the LRN conjecture is equivalent to the existence of S-units satisfying a variety of explicit conditions; in particular, it implies a refinement of Solomon's main result (op. cit.) on the $\mathfrak{P}$-adic valuation of Solomon's element ([BB2], 3.2 and 4.14). Moreover, the proof of the ETN conjecture for abelian fields proposed recently by [BG] uses Iwasawa co-descent in its final step and, in the splitting case, Solomon's elements appear again ([BG], 8.7, 8.11, 8.12, 8.13), together with the usual heavy calculations (see also [RW2]). Actually, all the technicalties seem to be related to the phenomenon of "trivial zeroes" of p-adic L-functions (with an intended vagueness, this means : zeroes of local Euler factors at p). This renders our present approach all the more interesting, since it appears to give a functorial (as opposed to technical) process to "bypass trivial zeroes". We hope to come back to this topic in a subsequent work.

1 A few exact sequences

In this section (which can be skipped at first reading), we collect for the convenience of the reader a few "well known" exact sequences which come from class-field theory and Iwasawa theory, and will be used freely in the rest of the paper. Let F be a number field, p an odd prime, $S = S(F)$ the set of primes in F which divide p. We 'll adopt once and for all the following notations :

U_F (resp. U'_F) = the group of units (resp. S-units) of F.

X_F (resp. X'_F) = the p-group of ideal classes (resp. S-ideal classes) of F.

$\mathfrak{X}_F$ = the Galois group over F of the maximal S-ramified (i.e. unramified outside S) abelian pro-p-extension of F.

For any abelian group, we'll denote by $\bar{A} = \varprojlim_m A/A^{p^m}$ the p-completion of A. If A is of finite type, then $\bar{A} = A \otimes \mathbb{Z}_p$.

1.1 Exact sequences from class-field theory

By class-field theory, X_F (resp. X'_F) is canonically isomorphic to the Galois group over F of the maximal unramified (resp. unramified and S-decomposed) abelian extension H_F (resp. H'_F) of F. We have two exact sequences :

– one relative to inertia :

$$\overline{U}_F \xrightarrow{\text{diag}} \mathcal{U} := \bigoplus_{v\in S} U_v^1 \xrightarrow{\text{Artin}} \mathfrak{X}_F \longrightarrow X_F \longrightarrow 0 \qquad (1)$$

Here $U_v^1 = \overline{U}_{F_v}$ is the group of principal local units at v, and "diag" is induced by the diagonal map. Assuming Leopoldt's conjecture for F and p (which holds e.g. when F is abelian), it is well known that "diag" is injective.

– one relative to decomposition :

$$\overline{U}'_F \xrightarrow{\text{diag}} \mathcal{F} := \bigoplus_{v\in S} \overline{F_v^\times} \xrightarrow{\text{Artin}} \mathfrak{X}_F \longrightarrow X'_F \longrightarrow 0 \qquad (2)$$

Here again, the map "diag" is injective modulo Leopoldt's conjecture. By putting together (1) and (2) (or by direct computation), we get a third exact sequence :

$$0 \longrightarrow \overline{U}'_F/\overline{U}_F \xrightarrow{\mu} \mathcal{F}/\mathcal{U} \xrightarrow{\text{Artin}} X_F \longrightarrow X'_F \longrightarrow 0 \tag{3}$$

where μ is induced by the valuation maps at all $v \in S$. By local class field theory $\overline{F_v^\times}/U_v^1$ is canonically isomorphic to the Galois group over F_v of the unramified $\mathbb{Z}_p$-extension of F_v. If F is Galois over $\mathbb{Q}$, then $\mathcal{F}/\mathcal{U} \simeq \mathbb{Z}_p[S]$ as Galois modules, and we can rewrite (3) as :

$$0 \longrightarrow \overline{U}'_F/\overline{U}_F \xrightarrow{\mu} \mathbb{Z}_p[S] \xrightarrow{\text{Artin}} X_F \longrightarrow X'_F \longrightarrow 0 \tag{3 μ}$$

1.2 Sinnott's exact sequence

For any $v \in S$, let $\widehat{F}_v^\times$ be the subgroup of $\overline{F_v^\times}$ which corresponds by class-field theory to the cyclotomic $\mathbb{Z}_p$-extension $F_{v,\infty}$ of F_v i.e. $\overline{F_v^\times}/\widehat{F}_v^\times \simeq \mathrm{Gal}(F_{v,\infty}/F_v)$ and $\widehat{F}_v^\times$ is usually called the group of *universal (cyclotomic) norms* of $F_{v,\infty}/F_v$. Let $\delta\colon \overline{U}'_F \longrightarrow \oplus_{v\in S}\overline{F_v^\times}/\widehat{F}_v^\times$ be the homomorphism induced by the diagonal map. Its kernel, $\widehat{U}'_F$ say, usually called the *Gross kernel* ([FGS]; see also [Kuz1]), consists of elements of $\overline{U}'_F$ which are everywhere universal cyclotomic norms. Its cokernel is described by the Sinnott exact sequence (see the appendix to [FGS]), for which we need additional notations.

Let $F_\infty = \bigcup_{n\geq 0} F_n$ be the cyclotomic $\mathbb{Z}_p$-extension of F, L'_∞ the maximal unramified abelian pro-p-extension of F_∞ which is totally decomposed at all places dividing p (hence at all places), L'_0 the maximal abelian extension of F contained in L'_∞. Then Sinnott's exact sequence reads :

$$0 \longrightarrow \overline{U}'_F/\widehat{U}'_F \xrightarrow{\delta} \bigoplus_{v\in S} \overline{F_v^\times}/\widehat{F}_v^\times \xrightarrow{\text{Artin}} \mathrm{Gal}(L'_0/F) \longrightarrow X'_F \longrightarrow 0 \tag{4}$$

By class-field theory, $\mathrm{Gal}(L'_\infty/F_\infty) \simeq X'_\infty := \varprojlim X'_{F_n}$ and $\mathrm{Gal}(L'_0/F_\infty) \simeq (X'_\infty)_\Gamma$ (the co-invariants of X'_∞ by $\Gamma = \mathrm{Gal}(F_\infty/F)$). By the product formula, the image of δ is contained readily in $\widetilde{\oplus}_{v\in S}\overline{F_v^\times}/\widehat{F}_v^\times$ (:= the kernel of the map "sum of components") and (4) can be rewritten as :

$$0 \longrightarrow \overline{U}'_F/\widehat{U}'_F \xrightarrow{\delta} \widetilde{\bigoplus}_{v\in S}\overline{F_v^\times}/\widehat{F}_v^\times \xrightarrow{\text{Artin}} (X'_\infty)_\Gamma \longrightarrow X'_F \tag{5}$$

The image of the natural map $(X'_\infty)_\Gamma \longrightarrow X'_F$ is nothing but $\mathrm{Gal}(H'_F/H'_F \cap F_\infty)$. Gross' conjecture asserts that $(X'_\infty)_\Gamma$ is finite. It holds (which is the case if F is abelian) if and only if $\widehat{U}'_F$ has $\mathbb{Z}_p$-rank equal to $r_1 + r_2$. Let us look more closely at the Galois structure of $\oplus_{v\in S}\overline{F_v^\times}/\widehat{F}_v^\times$. For simplicity, suppose that the completions of F are linearly disjoint from those of $\mathbb{Q}_\infty$ (this property should be called "local linear disjointness") : this happens e.g. if p is totally split in F. Then the local norm map gives an isomorphism $\overline{F_v^\times}/\widehat{F_v^\times} \simeq \overline{\mathbb{Q}_p^\times}/\widehat{\mathbb{Q}}_p^\times$; besides $\overline{\mathbb{Q}_p^\times}/\widehat{\mathbb{Q}}_p^\times \simeq 1 + p\mathbb{Z}_p$ is isomorphic to $p\mathbb{Z}_p$ via the p-adic logarithm, hence $\oplus_{v\in S}\overline{F_v^\times}/\widehat{F}_v^\times \simeq p\mathbb{Z}_p[S]$ in this situation. Let us denote by $I(S)$ the kernel of the sum in $\mathbb{Z}_p[S]$. Assuming that p is totally split in F, we can rewrite (5) as :

$$0 \longrightarrow \overline{U}'_F/\widehat{U}'_F \xrightarrow{\lambda} pI(S) \longrightarrow (X'_\infty)_\Gamma \longrightarrow X'_F \longrightarrow 0 \qquad (5\ \lambda)$$

On comparing the exact sequences (3 μ) and (5 λ), it is tempting to try and "draw a bridge" between them, but unfortunately, there seems to be no natural link between $\overline{U}'_F/\widehat{U}'_F$ and $\overline{U}'_F/\overline{U}_F$ because $\widehat{U}'_F$ and $\overline{U}_F$ are "independent" in the following sense :

Lemma 1.1 *Suppose that F satisfies Leopoldt's conjecture and p is totally split in F. Then*

$$\widehat{U}'_F \cap \overline{U}_F = \{1\}$$

Proof. Assuming Leopoldt's conjecture, we can embed $\overline{U}_F$ and $\widehat{U}'_F$ in $\mathcal{F} = \oplus_{v\in S}\overline{F_v^\times}$. If p is totally split, then for any $v \in S$, $\widehat{F}_v^\times = \widehat{\mathbb{Q}}_p^\times = p^{\mathbb{Z}_p}$ and $U_v^1 = 1 + p\mathbb{Z}_p$, hence an element $u \in \overline{U}_F$ belongs to $\widehat{U}'_F$ if and only if $u = 1$.
□

Nevertheless, one of our main results will be the construction of a natural map (§ 4) between two appropriate submodules of $\overline{U}'_F/\widehat{U}'_F$ and $\overline{U}'_F/\overline{U}_F$.

2 Hilbert's theorem 90 in Iwasawa theory

2.1 Some freeness results

We keep the notations of § 1. Let $\Lambda = \mathbb{Z}_p[[\Gamma]]$ be the Iwasawa algebra. If $M_\infty = \varprojlim M_n$ is a Λ-module, $\alpha_\infty = (\alpha_0, ..., \alpha_n, ...)$ will denote a typical element of M_∞, the index zero referring to $F = F_0$. The canonical map $(M_\infty)_\Gamma \longrightarrow M_0$ sends $\alpha_\infty + I_\Gamma M_\infty$ (where I_Γ is the augmentation ideal) to the component α_0 of α_∞. Let us denote by $M_\infty^{(0)}$ the submodule of M_∞ consisting of the elements α_∞ such that $\alpha_0 = 0$ (in additive notation). We want to study the relations between the Λ-modules $\overline{U}_\infty := \varprojlim \overline{U}_n$ and $\overline{U}'_\infty := \varprojlim \overline{U}'_n$. Obviously, they have the same Λ-torsion, which is $\mathbb{Z}_p(1)$ or (0) according as F contains or not a primitive p^{th} root of unity (see e.g. [Kuz1] or [Wi]). Let $\mathrm{Fr}_\Lambda(\overline{U}_\infty) := \overline{U}_\infty/\mathrm{Tor}_\Lambda(\overline{U}_\infty)$ and $\mathrm{Fr}_\Lambda(\overline{U}'_\infty) := \overline{U}'_\infty/\mathrm{Tor}_\Lambda(\overline{U}'_\infty)$.

Proposition 2.1 *Let F be any number field, $[F : \mathbb{Q}] = r_1 + 2r_2$. Then the Λ-modules $\mathrm{Fr}_\Lambda(\overline{U}_\infty)$ and $\mathrm{Fr}_\Lambda(\overline{U}'_\infty)$ are free, with Λ-rank equal to $r_1 + r_2$.*

Proof. The assertion concerning $\mathrm{Fr}_\Lambda(\overline{U}'_\infty)$ is a theorem of Kuz′min ([Kuz1], 7.2). As for $\mathrm{Fr}_\Lambda(\overline{U}_\infty)$, it is enough to notice that the quotient $\mathrm{Fr}_\Lambda(\overline{U}'_\infty)/\,\mathrm{Fr}_\Lambda(\overline{U}_\infty) \simeq \overline{U}'_\infty/\overline{U}_\infty$ has no $\mathbb{Z}_p$-torsion, hence has no non-trivial finite submodule. This is equivalent to saying that $\mathrm{Fr}_\Lambda(\overline{U}'_\infty)/\,\mathrm{Fr}_\Lambda(\overline{U}_\infty)$ is of projective dimension at most 1 over Λ (see e.g. [N1]). Since Λ is local, the Λ-freeness of $\mathrm{Fr}_\Lambda(\overline{U}_\infty)$ follows by Schanuel's lemma.
□

2.2 Cyclotomic submodules

If the field F is abelian over $\mathbb{Q}$, we have at our disposal the group C_F (resp. C'_F) of *Sinnott's circular units* (resp. *circular S-units*), which is defined as being the intersection of U_F (resp. U'_F) with the group of *circular numbers* of F ([Si1], [Si2]; see also §4.3 below). Sinnott's index formula states that $(U_F : C_F) = c_F h_F^+$, where h_F^+ is the class number of the maximal real subfield of F, and c_F is a rational constant whose explicit definition does not involve the class group. Going up the cyclotomic $\mathbb{Z}_p$-extension $F_\infty = \bigcup_n F_n$, it is known that the constants c_{F_n} remain bounded. It is a classical result (see e.g. [Gr1]) that $\overline{C}_\infty$ and $\overline{U}_\infty$ have the same Λ-rank. It is an obvious consequence of the definition that $\overline{C}_\infty$ and $\overline{C}'_\infty$ have the same Λ-rank. If the base field F is a cyclotomic field, it is also known, by results on distributions "à la Kubert-Lang", that $\mathrm{Fr}_\Lambda(\overline{C}_\infty)$ and $\mathrm{Fr}_\Lambda(\overline{C}'_\infty)$ are Λ-free (see e.g. [Kuz2]). Life would be too easy if such results could be extended to any abelian field. By proposition 2.1 and the exact sequences $0 \longrightarrow \mathrm{Fr}_\Lambda(\overline{C}_\infty) \longrightarrow \mathrm{Fr}_\Lambda(\overline{U}_\infty) \longrightarrow \overline{U}_\infty/\overline{C}_\infty \longrightarrow 0$, and $0 \longrightarrow \mathrm{Fr}_\Lambda(\overline{C}'_\infty) \longrightarrow \mathrm{Fr}_\Lambda(\overline{U}'_\infty) \longrightarrow \overline{U}'_\infty/\overline{C}'_\infty \longrightarrow 0$, we see that the Λ-freeness of $\mathrm{Fr}_\Lambda(\overline{C}_\infty)$ (resp $\mathrm{Fr}_\Lambda(\overline{C}'_\infty)$) is equivalent to the triviality of the maximal finite submodule of $\overline{U}_\infty/\overline{C}_\infty$ (resp. $\overline{U}'_\infty/\overline{C}'_\infty$). Let us call MF_∞ (resp. MF'_∞) these maximal finite submodules. In order to get hold of MF_∞ and MF'_∞, let us define :

Definition 2.2 *The* Kučera-Nekovář kernel *is defined by*

$$KN_{F,n} := \bigcup_{m \geq n} \mathrm{Ker}\, i_{n,m}$$

where for $m \geq n \in \mathbb{N}$, $i_{n,m} : \overline{U}_n/\overline{C}_n \longrightarrow \overline{U}_m/\overline{C}_m$ is the natural map.

The orders of the kernels $KN_{F,n}$ are bounded independently of n : this is an immediate consequence of the main result of [GK] and the one of [KN]. Let us denote by KN_F the projective limit (relatively to norm maps and n) $KN_F := \varprojlim KN_{F,n}$. This kernel KN_F is the obstruction to the Λ-freeness of $\mathrm{Fr}(\overline{C}_\infty)$ and $\mathrm{Fr}(\overline{C}'_\infty)$, in the following sense :

Proposition 2.3

(*i*) *$(\overline{U}_\infty/\overline{C}_\infty)^\Gamma$ is finite.*

(*ii*) *We have equalities $KN_F = MF_\infty = MF'_\infty$, and canonical isomorphisms $(KN_F)^\Gamma \simeq \mathrm{Tor}_{\mathbb{Z}_p}(\mathrm{Fr}(\overline{C}_\infty))_\Gamma \simeq \mathrm{Tor}_{\mathbb{Z}_p}(\mathrm{Fr}(\overline{C}'_\infty))_\Gamma$. In particular, $\mathrm{Fr}_\Lambda(\overline{C}_\infty)$ (resp. $\mathrm{Fr}_\Lambda(\overline{C}'_\infty)$) is free if and only if $KN_F = 0$.*

Proof. (*i*) The Main Conjecture (or Mazur-Wiles' theorem) applied to the maximal real subfield F^+ of F implies that the Λ-torsion modules $\overline{U}^+_\infty/\overline{C}^+_\infty = \overline{U}_\infty/\overline{C}_\infty$ and X^+_∞ have the same characteristic series (where the $^+$ denotes the objects related to F^+, i.e. the submodule on which complex conjugation acts trivially because F is abelian and $p \neq 2$). Because Leopoldt's conjecture holds all along the cyclotomic tower, it is well known that X_∞^+ and $(X'_\infty)^+$ are pseudo-isomorphic (see e.g. [Wi]). By Gross ' conjecture (which holds because F is abelian), $((X'_\infty)^+)^\Gamma$ is finite, and so is $(\overline{U}_\infty/\overline{C}_\infty)^\Gamma$.

(ii) We have an exact sequence :

$$0 \longrightarrow \overline{U}_\infty/\overline{C}_\infty \longrightarrow \overline{U}'_\infty/\overline{C}'_\infty \longrightarrow \overline{U}'_\infty/(\overline{U}_\infty + \overline{C}'_\infty) \longrightarrow 0$$

It gives an inclusion $MF_\infty \subset MF'_\infty$. To prove the inverse inclusion it is enough to show that $\overline{U}'_\infty/(\overline{U}_\infty + \overline{C}'_\infty)$ is without $\mathbb{Z}_p$-torsion. Let $\mathcal{S}$ be the (finite) set of places above p of F_∞. Clearly $\overline{U}'_\infty/\overline{U}_\infty$ is isomorphic to a submodule of finite index of $\mathbb{Z}_p[\mathcal{S}]$ with the natural action of $\mathrm{Gal}(F_\infty/\mathbb{Q})$ on both sides. Call M this submodule. It follows that $\overline{U}'_\infty/(\overline{U}_\infty + \overline{C}'_\infty)$ is isomorphic to $M/\mu(\overline{C}'_\infty)$, where μ is obtained from the valuations at finite levels, i.e. is the limit of the homomorphisms μ of the exact sequences (3).

Let $\mathbb{B}_\infty$ be the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, and $\mathbb{B}_n$ its n^{th} step. Let $\varepsilon_{\mathbb{Q},\infty}$ denotes the norm coherent sequence of numbers $\varepsilon_{\mathbb{Q},\infty} = (p, N_{\mathbb{Q}(\zeta_{p^2})/\mathbb{B}_1}(1-\zeta_{p^2}), \ldots, N_{\mathbb{Q}(\zeta_{p^{n+1}})/\mathbb{B}_n}(1-\zeta_{p^{n+1}}), \ldots)$. Since F_∞ is abelian over $\mathbb{Q}$ it is at most tamely ramified over $\mathbb{B}_\infty$, with ramification index e say. It follows that $\mu(\varepsilon_{\mathbb{Q},\infty}) = e\sum_{v\in\mathcal{S}} v$, with $e \in \mathbb{N}$ dividing $(p-1)$ and therefore a unit in $\mathbb{Z}_p$. Each generators of $\overline{C}_\infty$ either is a unit or behaves like $\varepsilon_{\mathbb{Q},\infty}$. This shows that $\mu(\overline{C}'_\infty) = \mathbb{Z}_p\sum_{v\in\mathcal{S}} v$. Consequently we have $\overline{U}'_\infty/(\overline{U}_\infty + \overline{C}'_\infty) \simeq M/(\mathbb{Z}_p\sum_{v\in\mathcal{S}} v) \subset \mathbb{Z}_p[\mathcal{S}]/(\mathbb{Z}_p\sum_{v\in\mathcal{S}} v)$ and this proves the second equality of proposition 2.3 (ii).

For the first equality, there exist an n_0 such that for all $n \geq n_0$, γ^{p^n} acts trivially on MF_∞. Putting $\omega_n = \gamma^{p^n} - 1$ as usual, and taking $n \geq n_0$ and $m-n$ large enough in the commutative triangle :

$$\begin{tikzcd}
\overline{U}_m/\overline{C}_m \arrow[rr, "\omega_m/\omega_n"] \arrow[dr, "N_{m,n}"'] & & \overline{U}_m/\overline{C}_m \\
& \overline{U}_n/\overline{C}_n \arrow[ur, "i_{n,m}"'] &
\end{tikzcd}$$

where $N_{m,n}$ is the obvious norm map, we see immediately that $MF_\infty \subset KN_F$, and the finiteness of KN_F shows the equality. Moreover, the exact sequence of Λ-modules

$$0 \longrightarrow \mathrm{Fr}_\Lambda(\overline{C}_\infty) \longrightarrow \mathrm{Fr}_\Lambda(\overline{U}_\infty) \longrightarrow \overline{U}_\infty/\overline{C}_\infty \longrightarrow 0$$

gives by descent an exact sequence of $\mathbb{Z}_p$-modules :

$$0 \longrightarrow (\overline{U}_\infty/\overline{C}_\infty)^\Gamma \longrightarrow \mathrm{Fr}_\Lambda(\overline{C}_\infty)_\Gamma \longrightarrow \mathrm{Fr}_\Lambda(\overline{U}_\infty)_\Gamma \longrightarrow (\overline{U}_\infty/\overline{C}_\infty)_\Gamma \longrightarrow 0$$

By (i) $(\overline{U}_\infty/\overline{C}_\infty)^\Gamma$ is finite, hence the equality $(\overline{U}_\infty/\overline{C}_\infty)^\Gamma = (MF_\infty)^\Gamma$. It follows then from 2.1 that $(MF_\infty)^\Gamma = \mathrm{Tor}_{\mathbb{Z}_p}(\mathrm{Fr}_\Lambda(\overline{C}_\infty)_\Gamma)$. The remaining assertions in 2.3 are straightforward.

□

In what follows the finite order $\#(KN_F)^\Gamma$ will play for descent modules a role analogous to the one played by Sinnott's constant c_F at finite level. Let us put the :

Notation 2.4 $\kappa_F := \#(KN_F)^\Gamma$

> *Remark :* We take this opportunity to correct a few mistakes in §3 and §4 of [BN] (due to the eventual non triviality of the constant κ_F). In short : every statement concerning only pseudo-isomorphisms or characteristic series remains true; every index formula should be corrected if necessary by a factor involving κ_F; in every monomorphism or isomorphism statement related to $(\overline{C}_\infty)_\Gamma$, this module should be replaced by its image in $\overline{U}_F$. Another, quicker solution would be to restrict generality and suppose everywhere that $\kappa_F = 1$.

In many situations (e.g.when $p \nmid [F : \mathbb{Q}]$, or when F is a cyclotomic field), it is known that $KN_F = 0$, but note that this is indirect evidence, coming from the Λ-freeness of $\mathrm{Fr}_\Lambda(\overline{C}_\infty)$ ([Kuz2], [B1] ...), not from the definition of the obstruction kernels. Let us describe examples of non trivial KN_F (again by indirect evidence, this time finding a $(\overline{C}_\infty)_\Gamma$ containing non trivial $\mathbb{Z}_p$-torsion). Note that another example for $p = 3$ has just been announced by R. Kučera ([Kuč]). Such examples may be considered as exceptional : let us recall that we have to avoid *Hypothèse B* of [B1], which holds true in most cases. In order to prove the non-triviality of the constant κ_F we need to add some very peculiar decomposition hypotheses such as those used in [Gr2]. Then the Galois module structure of circular units is less difficult to control. This justifies the terminology *"günstige $(p+1)$-tuple"* used in [Gr2]. We state these conditions :

1– the conductor f of F is of the form $f = \prod_{i=1}^{i=p+1} l_i$, where l_i are prime numbers, $l_i \equiv 1[p]$, and for all $j \neq i$, there exist some $x_{i,j}$ such that $l_i \equiv x_{i,j}^p [l_j]$.

2– $G := \mathrm{Gal}(F/\mathbb{Q}) \simeq (\mathbb{Z}/p\mathbb{Z})^2$

3– All the $(p+1)$ subfields of absolute degree p of F ($F^1, F^2, ..., F^{p+1}$ say) have conductors $\mathrm{cond}(F^j) = \prod_{i=1, i\neq j}^{i=p+1} l_i$

> *Remark :* Following Greither, and using Čebotarev density theorem, it is not difficult to prove that there exist infinitely many $(p+1)$-uples of primes l_i such that p and any subfield of the cyclotomic field $\mathbb{Q}(\zeta_{\prod l_i})$ satisfy condition 1. Such $(p+1)$-uples were called *günstige $(p+1)$-tuple* in [Gr2]. We can then deduce that for each p there exist (infinitely) many fields F such that F and p satisfy condition 1–, 2– and 3–.

Proposition 2.5 *If the fixed field F together with the fixed prime $p \neq 2$ satisfy condition 1–, 2– and 3–, then $\mathrm{Fr}(\overline{C}_\infty)$ is not Λ-free.*

Proof. Details are given in [B2].
□

2.3 Dividing by T

Going back to the general (not necessarily abelian) situation, let us record a useful result of co-descent, due to Kuz′min :

Proposition 2.6 ([Kuz1], 7.3) *The natural map* $(\overline{U}'_\infty)_\Gamma \longrightarrow \overline{U}'_F$ *is injective.*

> *Remark :* For simple reasons of $\mathbb{Z}_p$-ranks, the analogous property for $\overline{U}_\infty$ does not hold. In other words the natural map $(\overline{U}_\infty)_\Gamma \longrightarrow (\overline{U}'_\infty)_\Gamma$ induced by the inclusion $\overline{U}_\infty \subset \overline{U}'_\infty$, is not injective.

We aim to replace this last inclusion by a more elaborate injection. To this end, we first prove an analog of Hilbert's theorem 90 in Iwasawa theory :

Theorem 2.7 *Recall that* $(\overline{U}'_\infty)^{(0)}$ *is defined as the kernel of the naive descent map* $\overline{U}'_\infty \longrightarrow \overline{U}'_0$. *Let us fix a topological generator* γ *of* $\Gamma = \mathrm{Gal}(F_\infty/F)$. *Then multiplication by* $(\gamma - 1)$ *gives an isomorphism of* Λ*-modules :*

$$\overline{U}'_\infty \xrightarrow{\sim} (\overline{U}'_\infty)^{(0)}.$$

Proof. Since Γ acts trivially on $\overline{U}'_0$, the image $(\gamma-1)\overline{U}'_\infty$ is obviously contained in $(\overline{U}'_\infty)^{(0)}$. Moreover, the kernel of $(\gamma-1)$ is Λ-torsion, but the Λ-torsion submodule of $\overline{U}'_\infty$ is (0) or $\mathbb{Z}_p(1)$, hence $(\gamma-1)$ is injective on $\overline{U}'_\infty$. It remains to show that $(\gamma-1)\overline{U}'_\infty = (\overline{U}'_\infty)^{(0)}$. To this end, consider the composite of natural injective maps :

$$(\overline{U}'_\infty)^{(0)}/(\gamma-1)\overline{U}'_\infty \hookrightarrow \overline{U}'_\infty/(\gamma-1)\overline{U}'_\infty = (\overline{U}'_\infty)_\Gamma \hookrightarrow \overline{U}'_F$$

(the injectivity on the left is obvious; on the right it follows from 2.6). By definition (see the beginning of this section), this composite map is null, which shows what we want.

□

Corollary 2.8 *Suppose that* F *satisfies Leopoldt's conjecture and* p *is totally split in* F. *Then multiplication by* $(\gamma-1)$ *gives an isomorphism of* Λ*-modules* $\overline{U}'_\infty \xrightarrow{\sim} \overline{U}_\infty$

Proof. Using 2.7, it remains only to show the equality $(\overline{U}'_\infty)^{(0)} = \overline{U}_\infty$. Let us show successively the mutual inclusions :

(i) If $\alpha_\infty \in (\overline{U}'_\infty)^{(0)}$, then by 2.7, α_∞ may be written as $\alpha_\infty = (\gamma-1)\beta_\infty$, with $\beta_\infty \in \overline{U}'_\infty$. Since γ acts trivially on the primes of F above p, it is obvious that $(\gamma-1)\beta_\infty \in \overline{U}_\infty$. Hence $(\overline{U}'_\infty)^{(0)} \subset \overline{U}_\infty$.

(ii) Let $u_\infty \in \overline{U}_\infty$. For all primes v above p and all integers n, we have $u_0 = N_{F_n/F}(u_n) = N_{F_{v,n}/F_v}(u_n)$, i.e. u_0 is actually an element of $\widehat{U}'_F$. But $\overline{U}_F \bigcap \widehat{U}'_F = \{1\}$ by 1.1. Hence $\overline{U}_\infty \subset (\overline{U}'_\infty)^{(0)}$.

□

We are now in a position to construct the map we were looking for.

Definition 2.9

(*i*) *Suppose that* F *satisfies Leopoldt's conjecture and* p *is totally split in* F. *Define*

$$\Theta\colon \overline{U}_\infty \xrightarrow{\sim} \overline{U}'_\infty \xrightarrow{nat.} \left(\overline{U}'_\infty\right)_\Gamma \hookrightarrow \overline{U}'_F.$$

The first isomorphism is the inverse of the one in 2.8; the last monomorphism is the one in 2.6. The definition of Θ *depends on the choice of* γ.

(*ii*) *Suppose that* F *is abelian over* $\mathbb{Q}$. *Then* $F_\infty/\mathbb{Q}$ *is also abelian, and each* F_n *contains* circular units *in the sense of Sinnott [Si2]. Let* C_n *be the group of circular units of* F_n, $\overline{C}_n = C_n \otimes \mathbb{Z}_p$, $\overline{C}_\infty = \varprojlim \overline{C}_n$. *If* p *is totally split in* F, *define* $\Theta_C\colon \overline{C}_\infty \longrightarrow \overline{U}'_F$ *as the restriction of* Θ *to* $\overline{C}_\infty$.

The homomorphism Θ_C gives us Solomon's "S-units" at a lower cost than in [So1]. In fact the choice of any interesting element in $\overline{C}_\infty$ would yield via Θ_C an interesting element in $\overline{U}'_F$. Let us fix once and for all the notation $(\zeta_n)_{n\in\mathbb{N}}$ for a system of primitive n^{th} roots of unity such that $\zeta_{mn}^m = \zeta_n$. Let f be the conductor of F. Consider the norm coherent sequence of circular units $\varepsilon_{F,\infty} := (1, N_{\mathbb{Q}(\zeta_{p^2f})/F_1}(1-\zeta_{p^2f}), \cdots, N_{\mathbb{Q}(\zeta_{p^{n+1}f})/F_n}(1-\zeta_{p^{n+1}f}), \cdots)_{n\in\mathbb{N}}$. It is an easy (but fastidious) consequence of the definition that $\Theta_C(\varepsilon_{F,\infty})$ is indeed equal to Solomon's "p-unit" $\kappa(F,c)$, as constructed in [So1]. We skip the calculations since they give no further information. Actually we don't need to show the equality between $\kappa(F,c)$ and $\Theta_C(\varepsilon_{F,\infty})$ because in the sequel, all properties of $\Theta_C(\varepsilon_{F,\infty})$ which will be used (e.g. 4.2) will be reproved.

3 Modified circular S-units and regulators

3.1 Modified circular S-units

From now on, unless otherwise stated, F will be an *abelian number field, and* p *will be totally split in* F. We aim to study the maps Θ and Θ_C.

Proposition 3.1 $\operatorname{Ker}\Theta = (\gamma-1)\overline{U}_\infty$ *and* $\operatorname{Im}\Theta \simeq (\overline{U}'_\infty)_\Gamma$. *In particular* $\operatorname{Im}\Theta$ *has* $\mathbb{Z}_p$*-rank* (r_1+r_2).

Proof. The isomorphism $\operatorname{Im}\Theta \simeq (\overline{U}'_\infty)_\Gamma$ follows obviously from the definition of Θ and 2.6. The value of the $\mathbb{Z}_p$-rank comes from 2.1. To compute $\operatorname{Ker}\Theta$, we can decompose Θ as $\overline{U}_\infty \overset{\text{nat}}{\twoheadrightarrow} (\overline{U}_\infty)_\Gamma \overset{2.8}{\xrightarrow{\sim}} (\overline{U}'_\infty)_\Gamma \overset{2.6}{\hookrightarrow} \overline{U}'_F$. It is then obvious that $\operatorname{Ker}\Theta = (\gamma-1)\overline{U}_\infty$.
□

It is naturally more difficult to get hold of Θ_C, which contains more arithmetical information.

Proposition 3.2 $\operatorname{Ker}\Theta_C = ((\gamma-1)\overline{U}_\infty)\cap\overline{C}_\infty$ *and (abusing notation)* $\operatorname{Im}\Theta_C \simeq (\overline{C}_\infty)_\Gamma/(KN_F)^\Gamma$. *In particular* $\operatorname{Im}\Theta_C$ *has* $\mathbb{Z}_p$*-rank* (r_1+r_2).

Proof. The equality $\operatorname{Ker}\Theta_C = (\gamma - 1)\overline{U}_\infty \cap \overline{C}_\infty$ is 3.1. It follows that

$$\operatorname{Ker}((\overline{C}_\infty)_\Gamma \xrightarrow{\Theta_C} \overline{U}'_F) = \operatorname{Ker}((\overline{C}_\infty)_\Gamma \xrightarrow{\text{nat}} (\overline{U}_\infty)_\Gamma).$$

The latter has been shown in 2.3 to be the $\mathbb{Z}_p$ torsion of $(\overline{C}_\infty)_\Gamma$, isomorphic to $(KN_F)^\Gamma$.

□

Corollary 3.3 *Using 3.2 we define the map α by the commutative triangle :*

$$\begin{tikzcd}
(\overline{C}_\infty)_\Gamma \arrow[r, "\Theta_C"] \arrow[d, two heads] & \overline{U}'_F \\
(\overline{C}_\infty)_\Gamma/(KN_F)^\Gamma \arrow[ur, "\alpha"'] &
\end{tikzcd},$$

and the map $\beta\colon (\overline{C}_\infty)_\Gamma/(KN_F)^\Gamma \longrightarrow \overline{U}'_F/\overline{U}_F$ by composing α with the natural projection $\overline{U}'_F \twoheadrightarrow \overline{U}'_F/\overline{U}_F$. They both depend on γ and are injective.

Proof. The map α is well defined and injective by 3.2. For β, the image of α lies inside $\widehat{U}'_F$, and $\widehat{U}'_F \cap \overline{U}_F = \{1\}$ by lemma 1.1. Hence β is injective.

□

Definition 3.4 *Put $\widetilde{C}''_F := \operatorname{Im}\Theta_C = \operatorname{Im}\alpha$ and call it the subgroup of* modified circular S-units of F *(the terminology will be justified in 3.9 below). Solomon's "S-unit" is thus a particular modified circular S-unit.*

Obviously $\widetilde{C}''_F \subset (\overline{U}'_\infty)_\Gamma \subset \widehat{U}'_F \subset \overline{U}'_F$. By β in 3.3 we have $\widetilde{C}''_F = \operatorname{Im}\alpha \simeq \operatorname{Im}\beta$, and we may also consider $\widetilde{C}''_F$ as a subgroup of $\overline{U}'_F/\overline{U}_F$. The distinction between $\operatorname{Im}\alpha$ and $\operatorname{Im}\beta$ will always be made clear by the context.

Proposition 3.5 *$\widehat{U}'_F/\widetilde{C}''_F$ is the $\mathbb{Z}_p$-torsion of $\overline{U}'_F/\widetilde{C}''_F$, and its order is*

$$(\widehat{U}'_F : \widetilde{C}''_F) = \kappa_F \#((X'_\infty)^+)_\Gamma \ .$$

Proof. Because of the validity of Gross' conjecture, $\widehat{U}'_F$ has $\mathbb{Z}_p$-rank $(r_1 + r_2)$, hence $\widehat{U}'_F/\widetilde{C}''_F$ is $\mathbb{Z}_p$-torsion by 3.2. Besides, $\overline{U}'_F/\widehat{U}'_F$ is $\mathbb{Z}_p$-torsion free by Sinnott's exact sequence, hence the first part of the proposition. Let us compute the index $(\widehat{U}'_F : \widetilde{C}''_F) = (\widehat{U}'_F : (\overline{U}'_\infty)_\Gamma)((\overline{U}'_\infty)_\Gamma : \widetilde{C}''_F)$. Using 3.2 we have an isomorphism $(\overline{C}_\infty)_\Gamma/(KN_F)^\Gamma \xrightarrow{\sim} \widetilde{C}''_F$. Using the snake sequence of the proof of 2.3 (ii) (viz. applying the snake lemma to multiplication by $(\gamma - 1)$) we get an isomorphism $(\overline{U}'_\infty)_\Gamma/\widetilde{C}''_F \simeq (\overline{U}_\infty/\overline{C}_\infty)_\Gamma$. By a classical formula (Herbrand's quotient in Iwasawa theory), the order of the right hand side is p-adically equivalent to $\kappa_F G(0)$, where $G(T)$ is the common characteristic series of $(X'_\infty)^+$, X^+_∞ and $\overline{U}_\infty/\overline{C}_\infty$. In other terms $((\overline{U}'_\infty)_\Gamma : \widetilde{C}''_F) \overset{p}{\sim} \kappa_F\#((X'_\infty)^+)_\Gamma/\#((X'_\infty)^+)^\Gamma$. Note that by Gross' conjecture, $((X'_\infty)^-)^\Gamma$ is finite, hence null because $(X'_\infty)^-$ has no non-trivial finite submodule (see [I]). It follows that $((X'_\infty)^+)^\Gamma = (X'_\infty)^\Gamma$. As for the quotient $\widehat{U}'_F/(\overline{U}'_\infty)_\Gamma$, it is known to be isomorphic to $(X'_\infty)^\Gamma$ (see [Kuz1], 7.5). The proof of the proposition is complete.

□

3.2 Regulators

From now on, we impose the additional condition that F be *totally real*, in order to get regulator formulae. Note that in this case, all modules $\widetilde{C}''_F$, $(\overline{U}'_\infty)_\Gamma$, $\widehat{U}'_F$, and $\overline{U}'_F/\overline{U}_F$ have the same $\mathbb{Z}_p$-rank $r_1 = [F:\mathbb{Q}]$. Note also that in the totally real case, Leopoldt's conjecture (i.e. the finiteness of $(\mathfrak{X}_\infty)^\Gamma$, see e.g. [I] or [Wi]) implies what we called Gross' conjecture (i.e. the finiteness of $(X'_\infty)^\Gamma$) in §1.2. However we prefer to keep the terminology "Gross' conjecture" because in general the module $\mathfrak{X}_\infty$ and X'_∞ are not of the same nature : they are mutually in Spiegelung. Recall that we have injective homomorphisms (exact sequence (5 λ) and lemma 1.1) between $\mathbb{Z}_p$-lattices of rank $(r_1 - 1)$:

$$\overline{U}_F \hookrightarrow \overline{U}'_F/\widehat{U}'_F \overset{\lambda}{\hookrightarrow} pI(S)$$

We consider $pI(S)$ as a submodule of $I(S)$ and we define *regulators :*

Definition 3.6 *Since the abelian field F satisfies both conjectures of Leopoldt and Gross, we can define*

(i) R_F^{Gross} *as the index of* $\lambda(\overline{U}'_F/\widehat{U}'_F)$ *inside* $I(S)$

(ii) R_F^{Leop} *as the index of* $\lambda(\overline{U}_F)$ *inside* $I(S)$

Remarks

(i) The exact sequence (1) in §1.1 and the isomorphism $\mathcal{U}_F \simeq \bigoplus_{v\in S}(1+p\mathbb{Z}_p)$ (because p is totally split) show immediately that our index R_F^{Leop} is p-adically equivalent to the classical Leopoldt p-adic regulator. Note that we are obliged to work inside $I(S)$ (and not $pI(S)$) because of Leopoldt's definition.

(ii) The exact sequence (5) in §1.2 and the isomorphism $\oplus_{v\in S}\overline{F}_v^\times/\widehat{F}_v^\times \simeq \oplus_{v\in S}(1+p\mathbb{Z}_p)$ also show that, by taking p-adic logarithms, our index R_F^{Gross} is p-adically equivalent to a determinant which is the real analog of Gross' "imaginary" regulator as defined in [FGS].

(iii) Clearly, $R_F^{\mathrm{Gross}}/R_F^{\mathrm{Leop}}$ is an integer equal to $(\overline{U}'_F/\widehat{U}'_F : (\overline{U}_F \oplus \widehat{U}'_F)/\widehat{U}'_F) = (\overline{U}'_F : \overline{U}_F \oplus \widehat{U}'_F)$.

Theorem 3.7 *Let F be an abelian, totally real number field, and let p be totally split in F. Let h'_F be the S-class number of F. Then :*

(i) $\#\mathrm{Tor}_{\mathbb{Z}_p}(\overline{U}'_F/\widetilde{C}''_F) = (\widehat{U}'_F : \widetilde{C}''_F) \overset{p}{\sim} \kappa_F\ h'_F R_F^{Gross} p^{1-r_1}$

(ii) $(\overline{U}'_F/\overline{U}_F : \widetilde{C}''_F) = (\overline{U}'_F : \widetilde{C}''_F \oplus \overline{U}_F) \overset{p}{\sim} \kappa_F\ h'_F R_F^{Leop} p^{1-r_1}$

(the sign $\overset{p}{\sim}$ means p-adic equivalence).

Proof. The first equality in (i) has been shown in 3.5. It remains to compute the p-adic valuation of $\#(X'_\infty)_\Gamma$. From Sinnott's exact sequence (5 λ), we get : $\#(X'_\infty)_\Gamma \overset{p}{\sim} h'_F \#\mathrm{coker}\lambda$. This shows (i), and (ii) follows by the above calculation of $R_F^{\mathrm{Gross}}/R_F^{\mathrm{Leop}}$.
□

Corollary 3.8 *Let h_F be the class number of F. Then* $(\mathbb{Z}_p[S] : \mu(\widetilde{C}''_F)) \overset{p}{\sim} \kappa_F h_F R_F^{Leop} p^{1-r_1}$.

Proof. Recall that $\widetilde{C}''_F$ is embedded in $\overline{U}'_F/\overline{U}_F$ by 3.3, and the map μ takes place in the exact sequence :

$$0 \longrightarrow \overline{U}'_F/\overline{U}_F \xrightarrow{\mu} \mathbb{Z}_p[S] \xrightarrow{\mathrm{Artin}} X_F \longrightarrow X'_F \longrightarrow 0 \qquad (3\ \mu).$$

Together with 3.7 (ii), this shows the corollary.
□

Remark : In [So1], definition 4.1, Solomon introduces the Galois module $\mathcal{K}(F)$ generated by all Solomon elements attached to all subfields of F distinct from $\mathbb{Q}$, and he shows, in the semi-simple case, an index formula analogous to that of 3.8 (but he takes quotients by norms; see [So1], proposition 4.3). To compare $\overline{\mathcal{K}(F)}$ and $\widetilde{C}''_F$, see theorem 4.4 below. Corollary 3.8 is clearly a strengthening of Solomon's result. It gives the most general estimation of the size of the modified circular S-units. Note also that the regulator formula in 3.8 bears a resemblance with Leopoldt's formula giving the residue at 1 of the p-adic zeta function of F. This must (and will) be explained (see §5).

3.3 Kuz′min's modified circular S-units

Let us give now another description of the modified circular S-units. For simplicity, we stick to the hypotheses of theorem 3.7. We have seen that for the purpose of descent and co-descent in Iwasawa theory, it is often more convenient to use the S-units $\overline{U}'_\infty$ instead of the units $\overline{U}_\infty$ (for example : the natural map $(\overline{U}'_\infty)_\Gamma \longrightarrow \overline{U}'_F$ is injective by Kuz′min's theorem, here labelled 2.6, whereas $(\overline{U}_\infty)_\Gamma \longrightarrow \overline{U}_F$ is never injective). It is also natural to introduce the group C'_n of *circular S-units* of F_n in the sense of Sinnott, which is the intersection of U'_n with the *circular numbers* of F_n (for details see §4.3), and put $\overline{C}'_n = C'_n \otimes \mathbb{Z}_p$, $\overline{C}'_\infty = \varprojlim \overline{C}'_n$. The drawback is that the Λ-torsion module $\overline{U}'_\infty/\overline{C}'_\infty$ has no longer finite co-invariants (contrary to $\overline{U}_\infty/\overline{C}_\infty$). To get smoother descent, Kuz′min has introduced –without any splitting hypothesis on p– a certain module of *modified circular S-units at infinite level* which has been studied at length in [Kuz2] and [BN] (but note that [Kuz2] uses circular units in the sense of Washington – according to the terminology of [KN]– and [BN] in the sense of Sinnott). In our particular situation here (p totally split), Kuz′min's definition can be adapted and much simplified : let $\overline{C}''_\infty$ be the Λ-submodule of $\overline{U}'_\infty$ such that $(\overline{U}'_\infty/\overline{C}'_\infty)^\Gamma = \overline{C}''_\infty/\overline{C}'_\infty$. The obvious guess is then the good one :

Proposition 3.9 *With the hypotheses of theorem 3.7, the inclusion* $\overline{C}''_\infty \subset \overline{U}'_\infty$ *induces, by taking co-invariants, an isomorphism*

$$\widetilde{C}''_F \simeq (\overline{C}''_\infty)_\Gamma/(KN_F)^\Gamma$$

Proof. The proof will proceed in two steps :

Lemma 3.10 *Multiplication by* $(\gamma - 1)$ *induces isomorphisms of* Λ*-modules* $\overline{C}''_\infty \xrightarrow{\sim} \overline{C}_\infty$ *and* $\overline{U}'_\infty/\overline{C}''_\infty \cong \overline{U}_\infty/\overline{C}_\infty$.

Proof. Let τ be the inverse of the isomorphism $\overline{U}'_\infty \overset{\gamma-1}{\xrightarrow{\sim}} (\overline{U}'_\infty)^{(0)}$ of theorem 2.7. If $\alpha_\infty \in \tau(\overline{C}_\infty)$, then $(\gamma-1)\alpha_\infty \in \overline{C}_\infty \subset \overline{C}'_\infty$, hence $\alpha_\infty + \overline{C}'_\infty \in (\overline{U}'_\infty/\overline{C}'_\infty)^\Gamma$ and $\alpha_\infty \in \overline{C}''_\infty$. Conversely, if $\alpha_\infty \in \overline{C}''_\infty$, then $(\gamma-1)\alpha_\infty \in \overline{C}'_\infty$ by definition. But we have seen in the proof of 2.8 that $(\gamma-1)\alpha_\infty \in \overline{U}_\infty$, hence $(\gamma-1)\alpha_\infty \in \overline{C}_\infty$ i.e. $\alpha_\infty \in \tau(\overline{C}_\infty)$. (NB : we did not need to suppose F totally real).
$\square$

We now complete the proof of 3.9. By 3.10 and the construction of the map Θ, we see that $\widetilde{C}''_F$ is the image of $\overline{C}''_\infty$ by the natural map $\overline{U}'_\infty \to (\overline{U}'_\infty)_\Gamma \hookrightarrow \overline{U}'_F$. It remains only to show that the inclusion $\overline{C}''_\infty \subset \overline{U}'_\infty$ induces an injection $(\overline{C}''_\infty)_\Gamma/(KN_F)^\Gamma \hookrightarrow (\overline{U}'_\infty)_\Gamma$. This follows from 3.10 and 3.2.
$\square$

4 A bridge over troubled water

From now on, unless otherwise stated, F will be an *abelian number field, totally real and* p *will be totally split in* F, and let $G = \mathrm{Gal}(F/\mathbb{Q})$. Our goal is to compare the groups of circular units and modified circular S-units, $\overline{C}_F$ and $\widetilde{C}''_F$.

4.1 Statement of the problem

By construction $\widetilde{C}''_F \simeq ((\overline{C}_\infty)_\Gamma)/(KN_F)^\Gamma$ (proposition 3.9) but, as noticed in the introduction, the natural map $(\overline{C}_\infty)_\Gamma \longrightarrow \overline{C}_F$ gives no information. The idea is to replace it by the map $(\overline{C}_\infty)_\Gamma \longrightarrow \overline{U}'_F$ derived from Θ_C of 2.8, and to compare its image $\widetilde{C}''_F$ with the group of cyclotomic units $\overline{C}_F$ inside $\overline{U}'_F$. But staying in $\overline{U}'_F$ sheds no new light. Our strategy will be to consider $\widetilde{C}''_F$ inside $\overline{U}'_F/\overline{U}_F$ and $\overline{C}_F$ inside $\overline{U}'_F/\widehat{U}'_F$, and the to "lay a bridge over troubled water" between the exact sequences (3 μ) and (5 λ). As noticed in §1, the water is really troubled, because there is a priori no natural map between $\overline{U}'_F/\overline{U}_F$ and $\overline{U}'_F/\widehat{U}'_F$. For the sake of clarity, let us give first an abstract of the construction of the bridge :

- embed $\overline{C}_F$ in $\overline{U}'_F/\widehat{U}'_F$ by 1.1 and $\widetilde{C}''_F$ in $\overline{U}'_F/\overline{U}_F$ by 3.3
- then embed $\overline{C}_F$ in $I(S)$ via λ and $\widetilde{C}''_F$ in $\mathbb{Z}_p[S]$ via μ

– use, as in [So1], Coleman's theory of power series to compute the semi-local module $\mu(\widetilde{C}''_F)$

– this computation suggests to introduce an auxiliary module $\overline{\mathrm{Cyc}'}_F$ of "circular numbers", which contains naturally $\overline{C}_F$ and is sent naturally to $\widetilde{C}''_F$

– the composite map $\overline{C}_F \longrightarrow \overline{\mathrm{Cyc}'}_F \longrightarrow \widetilde{C}''_F$ gives what we want. Its kernel and cokernel are under control.

4.2 Explicit semi-local description of $\mu\left(\widetilde{C}''_F\right)$

We now review –and simplify– the proof of the main result of Solomon (see [So1], theorem 2.1). Let L be a local p-adic field, i.e. a finite extension of $\mathbb{Q}_p$. As previously, $\overline{L^\times}$ (resp. $\overline{U}(L)$) will denote the p-completion of $L^\times$ (resp of the units), and $\overline{L^\times_\infty} := \varprojlim \overline{L^\times_n}$, $\overline{U}_\infty(L) := \varprojlim \overline{U}(L_n)$ when going up the cyclotomic $\mathbb{Z}_p$-extension $L_\infty = \bigcup_{n\geq 0} L_n$ of L.

Lemma 4.1 (local analog of 2.7 and 2.8) *Suppose that $L/\mathbb{Q}_p$ is tamely ramified. Fix a topological generator γ of* $\mathrm{Gal}(L_\infty/L)$ *and let N be the norm map of $L/\mathbb{Q}_p$. Then multiplication by $(\gamma-1)$ induces an isomorphism*

$$\overline{\mathbb{Q}^\times_{p,\infty}} \overset{\gamma-1}{\underset{\sim}{\longrightarrow}} (\overline{\mathbb{Q}^\times_{p,\infty}})^{(0)} = N(\overline{U}_\infty(L)).$$

Proof. The first isomorphism (multiplying by T) is proved exactly in the same way as in 2.7. The second equality comes from the tameness assumption and the surjectivity of the norm map in this case.
□

We consider the special case $L = \mathbb{Q}_p(\zeta_p)$ and we want to make explicit the inverse of the isomorphism of 4.1, say $\tau_p\colon N(\overline{U}_\infty(L)) \overset{\sim}{\longrightarrow} \overline{\mathbb{Q}^\times_{p,\infty}}$. Let us start from the exact sequence of Λ-modules $0 \longrightarrow \overline{U}_\infty(L) \longrightarrow \overline{L^\times_\infty} \xrightarrow{v_\infty} \mathbb{Z}_p \longrightarrow 0$, where the valuation v_∞ is defined using a choice of norm coherent uniformizing elements $\pi_\infty = (\pi_n)$. This exact sequence in general does not split, but nevertheless, every element $x_\infty \in \overline{L^\times_\infty}$ may be uniquely written $x_\infty = z_\infty \pi_\infty^{v_\infty(x_\infty)}$, $z_\infty \in \overline{U}_\infty(L)$. In particular, take $x_\infty = \tau_p(N(u_\infty)) \in \overline{\mathbb{Q}^\times_{p,\infty}} \subset \overline{L^\times_\infty}$. By Coleman's theory over $\mathbb{Q}_p(\zeta_p)$ we can associate uniquely to u_∞ (resp. z_∞) formal power series $g_{u_\infty}(T)$ (resp. $g_{z_\infty}(T)$) in $\Lambda = \mathbb{Z}_p[[T]]$, $T = \gamma - 1$. Let c denotes the topological generator of $1+p\mathbb{Z}_p$ which corresponds to γ by the isomorphism $\mathbb{Z}_p^\times \cong \mathrm{Gal}(\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p)$. Then

Lemma 4.2 (Solomon's lemma) *(see [So1], Theorem 3.1).*
Let $\tau_p(N(u_\infty)) = z_\infty \pi_\infty^{a(u_\infty)}$. We have :

$$\prod_{\omega\in\mu_{p-1}} g_{u_\infty}((1+T)^\omega - 1) = \left(\frac{(1+T)^c-1}{T}\right)^{a(u_\infty)} \frac{g_{z_\infty}((1+T)^c-1)}{g_{z_\infty}(T)} \in \Lambda.$$

In particular $a(u_\infty) = (p-1)\log_c(g_{u_\infty}(0))$, *where* $\log_c(\cdot) = \log_p(\cdot)/\log_p(c)$, *and* $\log_p$ *is the Iwasawa logarithm characterized by* $\log_p(p) = 0$.

Proof. This is an immediate consequence of the construction of τ_p. The left hand side product is the norm of Coleman's power series $g_{u_\infty}(T)$. The right hand side is just obtained by applying $(\gamma - 1)$ to $(z_\infty \pi_\infty^{a(u_\infty)})$.
□

Let us now come back to our number field F and give an explicit description of the semi-local module $\mu(\widetilde{C}''_F) \subset \mathbb{Z}_p[S] = \oplus_{v|p}\mathbb{Z}_p v$. For any subfield M of F, let us denote by m the conductor of M (recall that $p \nmid m$). Consider the norm coherent sequence in the $\mathbb{Z}_p$-extension of M :

$$\varepsilon_{M,\infty} = \left(N_{\mathbb{Q}(\zeta_{mp^{n+1}})/M_n}(1 - \zeta_m^{\sigma^{-n}}\zeta_{p^{n+1}})\right)_{n\geq 0}$$

where $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_{f_M})/\mathbb{Q})$ is the Frobenius of p. For $M \neq \mathbb{Q}$, $\varepsilon_{M,\infty}$ is clearly an element of $\overline{C}_\infty(F)$. A system of Galois generators of $\overline{C}_\infty(F)$ is given by Greither's lemma ([Gr1], lemma 2.3) :

Lemma 4.3 *The elements* $\varepsilon_{M,\infty}$ *(for* $\mathbb{Q} \subsetneq M \subset F$*), together with* $(\gamma-1)\varepsilon_{\mathbb{Q},\infty}$*, form a system of* $\Lambda[G]$*-generators of* $\overline{C}_\infty(F)$*.*

By applying $\mu \circ \Theta$ to this system of generators we obtain a system of $\mathbb{Z}_p[G]$-generators of $\mu(\widetilde{C}''_F)$, consisting of the elements $\sum_v a(\imath_v(\varepsilon_{M,\infty}))v$, $\mathbb{Q} \subsetneq M \subset F$, $v \in S$ and of $\mu(p) = (1, ..., 1)$ (here $\imath_v$ is the embedding of F in F_v and $a(\cdot)$ is as in 4.2). We are then led by lemma 4.2 to evaluate at 0 the power series $g_{\varepsilon_{M,\infty}}(T)$. Once a choice of π_∞ has been made, these power series are uniquely determined. It is possible to choose a sequence π_∞ such that $g_{\varepsilon_{M,\infty}}(T)$ may be easily written down (actually $g_{\varepsilon_{M,\infty}}(T)$ will turn out to be a polynomial). It then gives $g_{\varepsilon_{M,\infty}}(0) = \imath_v(\varepsilon_M)$, where for all M, of conductor m say, ε_M is the cyclotomic number $\varepsilon_M := N_{\mathbb{Q}(\zeta_m)/M}(1-\zeta_m)$. Applying lemma 4.2 one gets $\mu(\Theta_C(\varepsilon_{M,\infty})) = (\log_c(\imath_v(\varepsilon_M)))_{v\in S}$. (for details see pp. 343 and 344 of [So1]). This proves the :

Theorem 4.4 *The* $\mathbb{Z}_p[G]$*-submodule* $\mu(\widetilde{C}''_F)$ *of* $\mathbb{Z}_p[S]$ *is generated by all the* $(\log_c(\imath_v(\varepsilon_M)))_{v\in S}$ *for* $\mathbb{Q} \subsetneq M \subset F$ *together with* $(1, ..., 1)$*, where* $\log_c(x)$ *denotes* $\log_p(x)/\log_p(c)$*.*
□

(compare with [So1], theorem 2.1; proposition 4.2)

Remark : Because of the injectivity of μ, this shows the precise relationship between our $\widetilde{C}''_F$ and Solomon's $\overline{\mathcal{K}(F)}$ ([So1], definition 4.1) :

$$\widetilde{C}''_F = \langle \mathcal{K}(F), p\rangle_{\mathbb{Z}_p}$$

4.3 Cyclotomic numbers

Let Cyc_F be the subgroup of cyclotomic numbers of $F^\times$ in the sense of Sinnott. More precisely, Cyc_F is generated by the elements ε_M (as defined in 4.4) and their Galois conjugates. These elements are actually local units except possibly at the primes of F which are ramified above $\mathbb{Q}$. Let us denote by $\mathrm{Ram}(F/\mathbb{Q})$ the set consisting of those primes. For technical reasons we'll consider Cyc_F as a subgroup of the group $U_F(R)$ of R-units of F, where $R = \mathrm{Ram}(F/\mathbb{Q}) \bigcup S \bigcup \{\text{primes of } F \text{ dividing } (1+p)\}$, and denote by $\overline{\mathrm{Cyc}_F}$ the closure of Cyc_F in $\overline{U}_F(R) = U_F(R) \otimes \mathbb{Z}_p$. We also introduce the group $\mathrm{Cyc}'_F = \langle \mathrm{Cyc}_F, (1+p) \rangle$ and its closure $\overline{\mathrm{Cyc}'}_F$. For any finite set of primes T containing S, the "T-analog" of Sinnott's exact sequence is valid, viz. we have :

$$\overline{U}_F(T) \xrightarrow{\delta_T} \bigoplus_{v\in T} \overline{F_v^\times}/\widehat{F_v^\times} \xrightarrow{\text{Artin}} \mathrm{Gal}(L'_0/F) \longrightarrow X_F(T) \longrightarrow 0 \tag{6}$$

Here $\widehat{F_v^\times}$, the *local cyclotomic norms*, is the group of norms in the $\mathbb{Z}_p$-extension $F_{v,\infty}/F_v$ (when $v \notin S$, $\widehat{F_v^\times}$ is the torsion of $\overline{F_v^\times}$, i.e. the local p-primary roots of unity). The exactness of (6) is obvious from (4), because L'_∞, hence L'_0, are independent of $T \supset S$. By Leopoldt's conjecture, $\overline{U}_F(T)$ is embedded in $\oplus_{v\in T}\overline{F_v^\times}$, and $\mathrm{Ker}\,\delta_T = \overline{U}_F(T) \cap \oplus_{v\in T}\widehat{F_v^\times}$. It follows at once from the definition of $\widehat{F_v^\times}$ that $ker\delta_T = \mathrm{Ker}\,\delta_S = \widehat{U}'_F$, independently of $T \supset S$.

> *Remark :* Jaulent's presentation of Sinnott's exact sequence (in [J]) enlarges T to the set of *all* primes of F. This has the advantage to dispense with Leopoldt's conjecture.

Theorem 4.5

(*i*) *We have a canonical epimorphism* $\overline{\mathrm{Cyc}'}_F \twoheadrightarrow \widetilde{C}''_F$.

(*ii*) *This epimorphism induces a monomorphism* $\varphi\colon \overline{C}_F \hookrightarrow {}_{\mathrm{Tr}}(\widetilde{C}''_F)$. *Here and from now on,* ${}_{\mathrm{Tr}}(\cdot)$ *will denote the kernel of the algebraic trace (= action of the trace element in* $\mathbb{Z}_p[G]$*).*

Proof. (*i*) Let λ_R be the composite map $(\oplus_{v|p}\log_p) \circ \mathrm{pr} \circ \delta_R$, where pr denotes the projection $\oplus_{v\in R}\overline{F_v^\times}/\widehat{F_v^\times} \twoheadrightarrow \oplus_{v\in S}\overline{F_v^\times}/\widehat{F_v^\times}$. By (3 μ) and (5 λ), we have a commutative square :

$$\begin{array}{ccc} \widetilde{C}''_F & \xrightarrow{\mu} & \mathbb{Z}_p[S] \\ \uparrow{\scriptstyle\phi} & & \uparrow{\scriptstyle\frac{1}{p}} \\ \overline{\mathrm{Cyc}'}_F & \xrightarrow{\lambda_R} & p\mathbb{Z}_p[S] \end{array}$$

where the dotted arrow ϕ exists because of 4.4 and the injectivity of μ. The description of the generators in 4.4 also ensures the surjectivity of ϕ.
(*ii*) By definition of λ_R, we have another commutative square

$$\begin{array}{ccc} \overline{\mathrm{Cyc}'}_F & \xrightarrow{\lambda_R} & p\mathbb{Z}_p[S] \\ \uparrow & & \| \\ \overline{C}_F & \xhookrightarrow{\lambda} & p\mathbb{Z}_p[S] \end{array}$$

Remark : Actually the image of λ is contained in $pI(S)$ because the norm of a (p)-unit is a power of p (up to a sign). But this holds no longer true for T-units, $T \supsetneq S$.

Gluing together the two commutative squares, we obtain still another :

$$\begin{array}{ccc} {}_{\mathrm{Tr}}\left(\widetilde{C}''_F\right) & \xhookrightarrow{\mu} & I(S) \\ \varphi \uparrow & & \wr \uparrow \frac{1}{p} \\ \overline{C}_F & \xhookrightarrow{\lambda} & pI(S), \end{array}$$

where φ is the restriction of ϕ to $\overline{C}_F$. The injectivity of φ follows from that of the three other sides of the square.

□

The monomorphism $\varphi\colon \overline{C}_F \hookrightarrow {}_{\mathrm{Tr}}(\widetilde{C}''_F)$ allows us to compare these two $\mathbb{Z}_p$-lattices in terms of ramification and "structural constants". Recall *Sinnott's index formula* : $(U_F : C_F) = c_F h_F$, where c_F is a rational constant whose explicit definition does not involve the class group of F, and has been extensively studied in [Si1], [Si2], [Kuz2], [BN], etc ... Recall that the cyclotomic constant c was introduced in 4.2, and κ_F in 2.4.

Theorem 4.6 *Let $p^b\mathbb{Z}_p$ be the ideal of $\mathbb{Z}_p$ generated by $[F:\mathbb{Q}]$ and by the numbers $[F:M_l]\log_c(l)$ where $M_l := F\cap\mathbb{Q}(\zeta_{l^\infty})$ and l runs through all prime divisors of the conductor of F such that $M_l \neq \mathbb{Q}$. Then $({}_{\mathrm{Tr}}(\widetilde{C}''_F) : \varphi(\overline{C}_F))$ is p-adically equivalent to $p^b c_F/\kappa_F$.*

Proof. By definition of φ, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{C}_F & \xrightarrow{\varphi} & {}_{\mathrm{Tr}}(\widetilde{C}''_F) & \longrightarrow & \mathrm{coker}\varphi & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \mu\log_p(c) & & \downarrow & & \\ 0 & \longrightarrow & \overline{U}_F & \xrightarrow{\lambda} & pI(S) & \longrightarrow & \mathrm{coker}\lambda & \longrightarrow & 0 \end{array}$$

and since $\log_p(c) \sim p$, we deduce equalities

$$\Big(I(S) : \mu({}_{\mathrm{Tr}}(\widetilde{C}''_F))\Big) = \big(pI(S) : \mathrm{Im}\log_p(c)\mu\big) = \big(\overline{U}_F : \overline{C}_F\big)\,\frac{\#\mathrm{coker}\lambda}{\#\mathrm{coker}\varphi}\ .$$

But $\#\mathrm{coker}\lambda \overset{p}{\sim} R_F^{\mathrm{Leop}} p^{1-[F:\mathbb{Q}]}$ by 3.6, and $(\overline{U}_F : \overline{C}_F) \overset{p}{\sim} c_F h_F$ by Sinnott's formula. It remains to compute the index $(I(S) : \mu({}_{\mathrm{Tr}}(\widetilde{C}''_F)))$. Using the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Tr}(\widetilde{C}''_F) \arrow[r] \arrow[d, hook, "\mu"'] & \widetilde{C}''_F \arrow[r, "\mathrm{Tr}"] \arrow[d, hook, "\mu"'] & \mathrm{Tr}(\widetilde{C}''_F) \arrow[r] \arrow[d, "\psi"'] & 0 \\ 0 \arrow[r] & I(S) \arrow[r] & \mathbb{Z}_p[S] \arrow[r, "\mathrm{Tr}"] & \mathbb{Z}_p \arrow[r] & 0 \end{tikzcd}$$

it appears that we just have to study the vertical map ψ. A priori we know that $\mathrm{Tr}(\widetilde{C}''_F)$ has $\mathbb{Z}_p$-rank one. But here we may identify the algebraic norm Tr with the arithmetic norm N from F down to $\mathbb{Q}$ (recall that we are dealing with S-units), so that $\mathrm{Tr}(\widetilde{C}''_F)$ may be identified with a submodule of $\overline{U}'_{\mathbb{Q}} = p^{\mathbb{Z}_p}$; in particular it has no $\mathbb{Z}_p$-torsion. This implies that $\operatorname{Ker}\psi = 0$. Concerning the image of ψ, the commutativity of the diagram shows that $\operatorname{Im}\psi$ is the ideal of $\mathbb{Z}_p$ generated by the elements $\sum_{v\in S} \log_c(\imath_v(\varepsilon_M)) = \log_c(N(\varepsilon_M))$, together with $[F:\mathbb{Q}] = \mathrm{Tr}(\mu(p))$. Even more explicitly, we know that if the conductor m of M is a power of a single prime, ℓ say, then $N(\varepsilon_M) = \ell^{[F:M]}$, and if m is composite, $N(\varepsilon_M) = 1$. Denoting (as indicated in the statement of the theorem) $\operatorname{Im}\psi = p^b\mathbb{Z}_p$, we have $(I(S) : \mu(\mathrm{Tr}(\widetilde{C}''_F))) = p^{-b}(\mathbb{Z}_p[S] : \mu(\widetilde{C}''_F))$ and this last quantity is p-adically equivalent to $\kappa_F p^{-b+1-r_1} h_F R_F^{\mathrm{Leop}}$ by corollary 3.8. The theorem is proved.

$\square$

Remark : In [So2], Solomon uses a description by generators and relations to compare (in our notations) $\overline{C}_F$ and $\widetilde{C}''_F$. In doing so, he must enlarge these modules to get (in his notations) a surjective map $\overline{\theta}\colon \overline{\mathcal{D}_{\mathcal{S}}} \twoheadrightarrow \overline{\mathcal{K}}_{\mathcal{S}}$ ([So2], theorem 1) which plays a role analogous to our map ϕ in the proof of 4.5. The restriction of $\overline{\theta}$ to $\overline{C}_F$ coincides with our map φ, and theorem 4.2 of [So2] determines its image. This construction by generators and relations allows one to compute $\mathbb{Z}_p$-ranks, but probably not indices nor annihilators.

5 Applications

5.1 Annihilation of class groups

As announced in the introduction, we are now looking for ideals of $\mathbb{Z}_p[G]$ which annihilate the p-groups X_F and $\overline{U}_F/\overline{C}_F$. Even more precisely, we intend to compute the Fitting ideals of some related modules. In order to state Solomon's conjecture, let us temporary relax the hypothesis that p is totally split in the totally real abelian field F. So now we only assume that p is a prime number which doesn't ramify in F, which in turn may be a real or complex abelian field. We fix an embedding $\imath\colon \mathbb{Q}^{\mathrm{sep}} \hookrightarrow \mathbb{Q}_p^{\mathrm{sep}}$, and we shall denote by $\mathcal{O}$ the topological closure in $\mathbb{C}_p$ of the image $\imath(O_F)$ of F (a priori $\mathcal{O}$ is inside $\mathbb{C}_p$, but it actually lies in $\mathbb{Q}_p^{\mathrm{sep}}$). For any subfield $M \neq \mathbb{Q}$ of F, we define *Solomon's element* sol_M as in [So1], §4 :

$$\mathrm{sol}_M := \frac{1}{p} \sum_{g\in \mathrm{Gal}(M/\mathbb{Q})} \left(\log_p(\imath(\varepsilon_M^g))\right) g^{-1} \in \mathcal{O}[\mathrm{Gal}(M/\mathbb{Q})].$$

By convention $\mathrm{sol}_{\mathbb{Q}} = 1$. These elements should be considered as real analogues of the classical Gauß sums, and the real analog of Stickelberger's theorem is *Solomon's conjecture* :

Conjecture 5.1 (see [So1], 4.1) sol_F *annihilates* $Cl_F \otimes \mathcal{O}$.

Solomon's conjecture holds true in the semi-simple case (i.e. $p \nmid [F : \mathbb{Q}]$), but this does not give anything new, because in this case it follows from Mazur-Wiles' theorem ([So1], 4.1; see also 6.8 below). In the general case, let us introduce a modified Solomon's element.

Definition 5.2 *Let F be a totally real abelian field. Recall that $w \mid p$ is the place corresponding to the fixed embedding $\imath\colon \mathbb{Q}^{sep} \hookrightarrow \mathbb{Q}_p^{sep}$. We still write w for the restriction of w to any subfield M of F.*

(i) Let M be any subfield of F. Let $\widetilde{\mathrm{sol}}_M^F$ be any element in $O_{F_w}[\mathrm{Gal}(F/\mathbb{Q})]$ which restricts to sol_M in $O_{M_w}[\mathrm{Gal}(M/\mathbb{Q})] \subset O_{F_w}[\mathrm{Gal}(M/\mathbb{Q})]$. In the commutative ring $O_{F_w}[\mathrm{Gal}(F/\mathbb{Q})]$, we define the product

$$\mathrm{sol}_M^F := \widetilde{\mathrm{sol}}_M^F \ \mathrm{Tr}_{F/M}.$$

(with obvious notation for the trace). This product does not depend on the choice of the lift $\widetilde{\mathrm{sol}}_M^F$.

(ii) For any real abelian field M with prime power conductor, let g_M be a generator of the (necessarily cyclic, M being real) group $\mathrm{Gal}(M/\mathbb{Q})$. For any subfield M of F we define

$$\mathrm{sol}_{M,2}^F := \begin{cases} (1-g_M)\,\mathrm{sol}_M^F & \text{if the conductor of } M \text{ is a prime power,} \\ \mathrm{sol}_M^F & \text{otherwise.} \end{cases}$$

We intend now to prove a slightly modified version of Solomon's conjecture in the non semi-simple case, with the additional hypothesis that p is totally split in F (then $\mathcal{O} = \mathbb{Z}_p$). It is here that our functorial approach will pay off, in that it will allow us to apply techniques "à la Rubin" to annihilate X_F :

Lemma 5.3 *Let N be a power of p. Let $V \subset \overline{U}_F/(\overline{U}_F)^N$ be a Galois submodule and let $\rho\colon V \longrightarrow \mathbb{Z}_p[G]/N\mathbb{Z}_p[G]$ be any equivariant homomorphism. Let $\mathcal{C} \subset U_F$ be the subgroup of "special units" as defined in* [R], p. 512, *and write $(\overline{\mathcal{C}})_N$ for its image $(\mathcal{C}(\overline{U}_F)^N)/(\overline{U}_F)^N \subset \overline{U}_F/(\overline{U}_F)^N$. Then $\rho(V \cap (\overline{\mathcal{C}})_N)$ annihilates X_F/NX_F.*

Proof. This lemma is a direct consequence in our special case of Rubin's theorem 1.3 (see [R]). The field denoted K in loc. cit. is here equal to $\mathbb{Q}$. We take for A of loc. cit. the full group X_F/NX_F. Then Rubin's theorem 1.3 shows that $\rho(V \cap \overline{\mathcal{C}})$ annihilates some submodule $A' \subset A$. Using lemma 1.6 (ii) and the definition of H_1 in loc. cit. it is easily checked that, in our special case (i.e. $p \neq 2$, the only roots of unity in F are ± 1, and every place above p is totally ramified in $F(\zeta_N)/F$), we actually have $A' = A = X_F/NX_F$.

□

Theorem 5.4 *For any totally real abelian field F, and any p totally split in F, $\mathrm{sol}_{F,2}^F$ annihilates X_F.*

N.B. : If the conductor f of F is divisible by at least two distinct primes this theorem is Solomon's conjecture 5.1, because by definition the elements $\mathrm{sol}_{F,2}^F$ and sol_F are equal. If the conductor is a power of a single prime ℓ this theorem is slightly weaker than Solomon's conjecture. But it is, for ideal classes, the perfect analogue of an anihilation result for unit classes that we shall prove later (see theorem 5.7).

Proof. Let $\rho_1 : \overline{U}_F \longrightarrow \mathbb{Z}_p[G]$ be the composite map $\rho_1 = \eta \circ \frac{1}{p} \circ \lambda$, where η is the isomorphism $\eta\colon \mathbb{Z}_p[S] = \mathbb{Z}_p[G]\ w \xrightarrow{\sim} \mathbb{Z}_p[G]$, and w is the prime corresponding to $\imath$. Fix N such that $(X_F/NX_F) = X_F$. We apply lemma 5.3 by taking $V = \overline{U}_F/(\overline{U}_F)^N$ and $\rho\colon V \longrightarrow \mathbb{Z}_p/N\mathbb{Z}_p[G]$ to be the map induced by ρ_1. Exactly as we defined $\mathrm{sol}_{F,2}^F$ we put :

$$\varepsilon_{F,2} := \begin{cases} (1-g_F)\varepsilon_F & \text{if the conductor of } F \text{ is a prime power,} \\ \varepsilon_F & \text{otherwise.} \end{cases}$$

Then it is a classical fact that $\varepsilon_{F,2}$ is a unit and the element $\mathrm{sol}_{F,2}^F$ is nothing else but the image $\rho_1(\varepsilon_{F,2})$. Further $\varepsilon_{F,2}$ is a special unit : in the special case that F is the maximal real subfield of a cyclotomic field this is theorem 2.1 of loc. cit., and for any abelian field the result for $\varepsilon_{F,2}$ follows from exactly the same computation. By lemma 5.3 it follows that the class in $\mathbb{Z}_p[G]/N$ of $\mathrm{sol}_{F,2}^2$ anihilates $X_F/NX_F = X_F$.

□

5.2 Fitting ideals of quotients of units

Theorem 5.4 is still unsatisfactory, in that it uses only the map λ, not the map μ (notation of §1). Let us cross the bridge built in 4.5. We'll rather work with ideals than with elements :

Definition 5.5 *We define ideals $\mathrm{Sol}_1(F) \supset \mathrm{Sol}_2(F)$ of $O_{F_w}[G]$ by giving sets of generators :*

$$\mathrm{Sol}_1(F) := \left\langle \mathrm{sol}_M^F \,|\mathbb{Q} \subseteq M \subseteq F \right\rangle \supset \mathrm{Sol}_2(F) := \left\langle \mathrm{sol}_{M,2}^F \,|\mathbb{Q} \subseteq M \subseteq F \right\rangle.$$

Proposition 5.6 *Suppose that p is totally split in F. Recall that η is the isomorphism $\eta\colon \mathbb{Z}_p[S] = \mathbb{Z}_p[G]\ w \xrightarrow{\sim} \mathbb{Z}_p[G]$. Then :*

$$\mathrm{Sol}_1(F) = \eta \circ \frac{1}{p} \circ \lambda_R(\overline{\mathrm{Cyc}'_F}) \text{ and } \mathrm{Sol}_2(F) = \eta \circ \frac{1}{p} \circ \lambda_R(\overline{C}_F \oplus (1+p)^{\mathbb{Z}_p}).$$

N.B. : The set of places R has been chosen in order that $\overline{U}_F(R)$ contains simultaneously Cyc'_F, $\overline{C}_F$ and $(1+p)^{\mathbb{Z}_p}$, the last two being direct summands.

Proof. By definition of the map λ_R, we have $\mathrm{sol}_M^F = \eta \circ \frac{1}{p} \circ \lambda_R(\varepsilon_M)$. Since these elements, together with $(1+p)$ (which is sent to $\mathrm{Tr}_{K/\mathbb{Q}}$, up to a p-adic unit) generate $\overline{\mathrm{Cyc}'_F}$, the first equality is proved. The second is proved in exactly the same way.

□

We are now in a position to prove global annihilation results for various quotients of units by circular units :

Theorem 5.7 *Let F be a totally real abelian field, let p be totally split in F. Then :*

(i) Let $\mathcal{U}_n$ be the semi-local module $\mathcal{U}_n := \oplus_{v\in S} U^1_v(F_n)$ and $\mathcal{U}_\infty := \varprojlim \mathcal{U}_n$. The embedding $\overline{C}_\infty \hookrightarrow \mathcal{U}_\infty$ induces, by taking co-invariants, an embedding $\widetilde{C}''_F \hookrightarrow (\mathcal{U}_\infty)_\Gamma$, and $\mathrm{Sol}_1(F)$ is the initial Fitting ideal of the $\mathbb{Z}_p[G]$-quotient module.

(ii) Let $\varphi\colon \overline{C}_F \hookrightarrow \widetilde{C}''_F$ be the map defined in 4.5, and extend it to a map $\widetilde{\varphi}\colon \overline{C}_F \oplus (1+p)^{\mathbb{Z}_p} \longrightarrow \widetilde{C}''_F$ by putting $\widetilde{\varphi}(1+p) = p$. Then $\mu\circ\widetilde{\varphi}$ gives an embedding $\overline{C}_F \oplus (1+p)^{\mathbb{Z}_p} \hookrightarrow (\mathcal{U}_\infty)_\Gamma$, and $\mathrm{Sol}_2(F)$ is the initial Fitting ideal of the $\mathbb{Z}_p[G]$-quotient module.

(iii) $\mathrm{Sol}_2(F)$ annihilates $\overline{U}_F/\overline{C}_F$.

Proof. (i) The analog of the exact sequence (1) at infinite level gives an injection $\mathcal{U}_\infty/\overline{U}_\infty \hookrightarrow \mathfrak{X}_\infty := \varprojlim \mathfrak{X}_{F_n}$. Leopoldt's conjecture is known to be equivalent to $\mathfrak{X}_\infty^\Gamma = 0$, hence implies $(\mathcal{U}_\infty/\overline{U}_\infty)^\Gamma = 0$. It follows that $(\overline{U}_\infty)_\Gamma \hookrightarrow (\mathcal{U}_\infty)_\Gamma$ and that $(\overline{C}_\infty)_\Gamma \longrightarrow (\mathcal{U}_\infty)_\Gamma$ has the same kernel as $(\overline{C}_\infty)_\Gamma \longrightarrow (\overline{U}_\infty)_\Gamma$, hence the embedding $\widetilde{C}''_F \hookrightarrow (\mathcal{U}_\infty)_\Gamma$. Let us introduce another semi-local module, namely $\mathcal{F}_\infty(F) := \varprojlim \overline{\oplus_{v\in S} F^\times_{n,v}}$. We have seen that multiplication by $(\gamma-1)$ induces an isomorphism $\overline{C}''_\infty \xrightarrow{\sim} \overline{C}_\infty$ (by 3.10) and analogously, $\mathcal{F}_\infty(F) \xrightarrow{\sim} N(\mathcal{U}_\infty(F(\zeta_p)))$ (by 4.1). But $N(\mathcal{U}_\infty(F(\zeta_p))) = \mathcal{U}_\infty(F)$ by tame ramification. In other words, we have a commutative square

$$\begin{tikzcd}
\overline{C}''_\infty \arrow[r, "\sim", "\gamma-1"'] \arrow[d, hook] & \overline{C}_\infty \arrow[d, hook] \\
\mathcal{F}_\infty(F) \arrow[r, "\sim", "\gamma-1"'] & \mathcal{U}_\infty(F) = \mathcal{U}_\infty
\end{tikzcd}$$

By codescent we then get a commutative diagram :

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{C}''_F = (\overline{C}''_\infty)_\Gamma/(KN_F)^\Gamma \arrow[r, "\mu"] \arrow[d, "\wr"] & \mathcal{F}_\infty(F)_\Gamma \cong \mathbb{Z}_p[S] \arrow[r] \arrow[d, "\wr"] & \mathrm{coker}\mu \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & (\overline{C}_\infty)_\Gamma/(KN_F)^\Gamma \arrow[r] & (\mathcal{U}_\infty)_\Gamma \arrow[r] & (\mathcal{U}_\infty/\overline{C}_\infty)_\Gamma \arrow[r] & 0
\end{tikzcd}$$

which shows that $(\mathcal{U}_\infty/\overline{C}_\infty)_\Gamma$ has the same $\mathbb{Z}_p[G]$-Fitting ideal as $\mathrm{coker}\mu$. By the isomorphism $\eta\colon \mathbb{Z}_p[S] \xrightarrow{\sim} \mathbb{Z}_p[G]$, this last Fitting is nothing but the ideal $\eta\circ\mu(\widetilde{C}''_F) = \eta\circ\frac{1}{p}\circ\lambda_R(\overline{\mathrm{Cyc}'}_F) = \mathrm{Sol}_1(F)$.

(ii) The commutative diagram

$$\begin{tikzcd}
\overline{C}_F \oplus (1+p)^{\mathbb{Z}_p} \arrow[r, hook, "\tilde{\varphi}"] \arrow[d, hook] & \widetilde{C}''_F \arrow[d, hook, "\mu"] \\
\overline{U}_F \oplus (1+p)^{\mathbb{Z}_p} \arrow[r, hook, "\frac{1}{p}\circ\lambda"] & \mathbb{Z}_p[S]
\end{tikzcd}$$

(see the construction of φ in 4.5) and the isomorphism $\mathbb{Z}_p[S] \simeq (\mathcal{U}_\infty)_\Gamma$ in (i) above show that $\mu \circ \tilde{\varphi}$ embeds $\overline{C}_F \oplus (1+p)^{\mathbb{Z}_p}$ into $(\mathcal{U}_\infty)_\Gamma$. We conclude by applying η as in (i).

(iii) It is well known that the Fitting ideal of a module is contained in its annihilator. Here $\mathrm{Sol}_2(F)$ will annihilate $(\mathcal{U}_\infty)_\Gamma/\mu \circ \tilde{\varphi}(\overline{C}_F \oplus (1+p)^{\mathbb{Z}_p})$, which contains, by the commutative square in (ii), an isomorphic image of $\overline{U}_F/\overline{C}_F$.

□

Corollary 5.8 *(Real analog of Stickelberger's index)*

$$(\mathbb{Z}_p[G] : \mathrm{Sol}_1(F)) = (\mathbb{Z}_p[S] : \mu(\widetilde{C}''_F)) = \kappa_F \#\mathrm{Tor}_{\mathbb{Z}_p}\mathfrak{X}_F \overset{p}{\sim} \kappa_F h_F R_F^{Leop} p^{1-r_1}$$

Proof. It has been shown in 5.7 (i) that $(\mathbb{Z}_p[G] : \mathrm{Sol}_1(F)) = (\mathbb{Z}_p[S] : \mu(\widetilde{C}''_F)) = \#(\mathcal{U}_\infty/\overline{C}_\infty)_\Gamma$. Since $\mathfrak{X}_\infty$ has no non-trivial Γ-invariants (Leopoldt's conjecture) and has the same characteristic series as $\mathcal{U}_\infty/\overline{C}_\infty$, we deduce the equivalence : $\#(\mathcal{U}_\infty/\overline{C}_\infty)_\Gamma \overset{p}{\sim} \#(\mathcal{U}_\infty/\overline{C}_\infty)^\Gamma \#(\mathfrak{X}_\infty)_\Gamma$. As F is totally real, $(\mathfrak{X}_\infty)_\Gamma$ is obviously isomorphic to $\mathrm{Tor}_{\mathbb{Z}_p}\mathfrak{X}_F$. It remains to compute the order of $\mathrm{Tor}_{\mathbb{Z}_p}\mathfrak{X}_F$: this a classical calculation using p-adic L-functions (see e.g. [BN], 2.6), which is actually equivalent to Leopoldt's p-adic formula (the discriminant does not appear here because p is unramified in F).

□

> *Remark :* This gives another proof of formula 3.8, at the same time explaining the parenthood between this index formula and Leopoldt's residue formula.

5.3 Still another exact sequence

Let us extract from the proof of 5.7 the following analog of the exact sequences (1) and (2) of class field theory :

Corollary 5.9 *We have an isomorphism $(\mathcal{U}_\infty/\overline{C}_\infty)_\Gamma \simeq (\oplus_{v\in S}\widehat{F}_v^\times)/\widetilde{C}''_F$ (notation of §1.2) and an exact sequence of finite modules :*

$$0 \longrightarrow \widehat{U}'_F/\widetilde{C}''_F \longrightarrow (\oplus_{v\in S}\widehat{F}_v^\times)/\widetilde{C}''_F \longrightarrow \mathrm{Tor}_{\mathbb{Z}_p}\mathfrak{X}_F \longrightarrow (X'_\infty)_\Gamma \longrightarrow 0 \qquad (7)$$

Proof. It has been shown in the proof of 5.7 that $(\mathcal{U}_\infty/\overline{C}_\infty)_\Gamma \simeq (\mathcal{F}_\infty(F))_\Gamma/\widetilde{C}''_F$, where $\mathcal{F}_\infty(F) := \varprojlim \oplus_{v\in S}\overline{F_{n,v}^\times}$ (notation of §1.2). But $(\overline{F_{v,\infty}^\times})_\Gamma = (\overline{F_{v,\infty}^\times})_{\Gamma_v} \simeq \widehat{F}_v^\times$ by class field theory. Thus the isomorphism is proved.

From the exact sequence :

$$\begin{tikzcd} 0 \arrow[r] & \overline{U}'_\infty \arrow[r] & \mathcal{F}_\infty(F) \arrow[rr] \arrow[dr, two heads] & & \mathfrak{X}_\infty \arrow[r] & X'_\infty \arrow[r] & 0 \\ & & & \mathcal{D}_\infty \arrow[ur, hook] & & & \end{tikzcd}$$

(where $\mathcal{D}_\infty$ denotes the relevant subgroup generated by decomposition subgroups), we get by co-descent two exact sequences :

$$0 \longrightarrow (\overline{U}'_\infty)_\Gamma \longrightarrow \oplus_{v\in S}\widehat{F}_v^\times \longrightarrow (\mathcal{D}_\infty)_\Gamma \longrightarrow 0 \quad \text{and}$$

$$0 \longrightarrow (X'_\infty)^\Gamma \longrightarrow (\mathcal{D}_\infty)_\Gamma \longrightarrow (\mathfrak{X}_\infty)_\Gamma \simeq \mathrm{Tor}_{\mathbb{Z}_p}\mathfrak{X}_F \longrightarrow (X'_\infty)_\Gamma \longrightarrow 0$$

Putting them together and using Kuz'min's isomorphism $\widehat{U}'_F/(\overline{U}'_\infty)_\Gamma \simeq (X'_\infty)^\Gamma$ (see [Kuz1], 7.5), we get the exact sequence (which is general) :

$$0 \longrightarrow \widehat{U}'_F \longrightarrow \oplus_{v\in S}\widehat{F}_v^\times \longrightarrow \mathrm{Tor}_{\mathbb{Z}_p}\mathfrak{X}_F \longrightarrow (X'_\infty)_\Gamma \longrightarrow 0 \text{, hence also (7).}$$

□

> *Remark :* In the study of the Main Conjecture via circular units inside semi-local units and Coleman's theory, and even in the semi-simple case, a problem arises for characters ψ of $G = \mathrm{Gal}(F/\mathbb{Q})$ such that $\psi(p) = 1$, because the natural co-descent map $(\mathcal{U}_\infty/\overline{C}_\infty)_{\Gamma,\psi} \longrightarrow (\mathcal{U}_F/\overline{C}_F)_\psi$ then gives no information (see e.g. [Gi]). This difficulty is related to the phenomenon of trivial zeroes of p-adic L-functions. In this context, the more "sophisticated" exact sequence (7) could be viewed as a device to "bypass trivial zeroes". We hope to come back to it in greater detail.

5.4 A criterion for Greenberg's conjecture

Greenberg's conjecture asserts that the orders of the groups $X_n = X_{F_n}$ are bounded in the cyclotomic $\mathbb{Z}_p$-extension of a totally real number field F. This conjecture has been numerically verified by various authors in a huge amount of special cases. The case when p splits in the base field F is notably more difficult than the others, probably because of the phenomenon of trivial zeroes. Our approach here gives a new proof of a known criterion (see [Ta], theorem 1.3; but note that Taya only needs Leopoldt's conjecture) :

Theorem 5.10 *Let F be a real abelian field and p be an odd prime totally split in F. Let $D_n \subset X_n$ be the submodule generated by (images of) the places of F_n above p. Then Greenberg's conjecture holds true for F and p if and only if $\#D_n = \#\mathrm{tor}_{\mathbb{Z}_p}(\mathfrak{X}_F)$ for every n sufficiently large.*

Proof. By definition D_n is the kernel $D_n = \mathrm{Ker}(X_n \longrightarrow X'_n)$. Hence it may be used to split the sequence (3 μ) in two shorter sequences. Taking projective limits we get :

$$0 \longrightarrow \overline{U}_\infty \longrightarrow \overline{U}'_\infty \xrightarrow{\mu} \mathbb{Z}_p[S] \longrightarrow D_\infty \longrightarrow 0 \qquad (3\mu\, a)$$

$$0 \longrightarrow D_\infty \longrightarrow X_\infty \longrightarrow X'_\infty \longrightarrow 0 \qquad (3\mu\, b)$$

Since F_∞/F is totally ramified at all places above p, D_∞ is a submodule of X_∞^Γ and therefore is finite. Again by total ramification, the maps $D_{n+1} \longrightarrow D_n$

are surjective. So the theorem asserts that Greenberg's conjecture is equivalent to the equality $\#D_\infty = \#\mathrm{tor}_{\mathbb{Z}_p}(\mathfrak{X}_F)$. From corollary 2.8 we have $\overline{U}'_\infty/\overline{U}_\infty = (\overline{U}'_\infty)_\Gamma$, hence by $(3\mu\, a)$ we get the isomorphism $D_\infty \cong \mathbb{Z}_p[S]/\mu((\overline{U}'_\infty)_\Gamma)$. Using corollary 3.8 we deduce the equality :

$$\#D_\infty(\mu((\overline{U}'_\infty)_\Gamma) : \mu(\widetilde{C}''_F)) = \#D_\infty((\overline{U}'_\infty)_\Gamma : \widetilde{C}''_F) = \kappa_F \#\mathrm{tor}_{\mathbb{Z}_p}(\mathfrak{X}_F) \qquad (\dagger)$$

Now Greenberg's conjecture is equivalent to the finiteness of $\overline{U}_\infty/\overline{C}_\infty$, which in turn is equivalent to the equality $\overline{U}_\infty/\overline{C}_\infty = KN_F$. Crossing the bridge, actually applying lemma 3.10, this equality becomes equivalent to the canonical isomorphism $KN_F \cong \overline{U}'_\infty/\overline{C}''_\infty$. By Nakayama's lemma, this isomorphism is equivalent to the equality $((\overline{U}'_\infty)_\Gamma : \widetilde{C}''_F) = \#(KN_F)_\Gamma$. Since KN_F is finite we already have the equalities $\#(KN_F)_\Gamma = \#(KN_F)^\Gamma = \kappa_F$. Comparison with the equality $(\dagger)$ shows the theorem.

□

For numerical applications of criterion 5.10, see [Ta] and the references therein.

6 Comparison of Fitting ideals

At this point, a natural question arises : does theorem 5.7 give anything new with respect to the Main Conjecture ? In this section, the general hypothesis will be : F is an *abelian totally real field* (and p is not necessarily totally split in F), $G = \mathrm{Gal}(F/\mathbb{Q})$. To avoid petty technical trouble, let us also suppose that *F is linearly disjoint from $\mathbb{Q}_\infty$* (which is the case if p is totally split in F), in order that G acts on all the natural modules attached to the fields F_n in the cyclotomic $\mathbb{Z}_p$-extension F_∞/F.

Definition 6.1 *Let $MW(F)$ (for Mazur-Wiles) be the initial Fitting ideal of $\mathrm{Tor}_{\mathbb{Z}_p}\mathfrak{X}_F$ over $\mathbb{Z}_p[G]$.*

This ideal annihilates $\mathrm{Tor}_{\mathbb{Z}_p}\mathfrak{X}_F$, hence also (since F is supposed linearly disjoint from $\mathbb{Q}_\infty$) the p-class group X_F, but we do not know a priori if it annihilates $\overline{U}_F/\overline{C}_F$. Conversely, if p is totally split in F, recall that the ideal $\mathrm{Sol}_2(F)$ annihilates both $\overline{U}_F/\overline{C}_F$ and X_F. We would like to compare $\mathrm{Sol}_1(F)$ and $MW(F)$.

At infinite level, class-field theory gives an exact sequence of $\Lambda[G]$-modules :

$$0 \longrightarrow \overline{U}_\infty/\overline{C}_\infty \longrightarrow \mathcal{U}_\infty/\overline{C}_\infty \longrightarrow \mathfrak{X}_\infty \longrightarrow X_\infty \longrightarrow 0$$

We get by co-descent and functoriality :

$$MW(F) = \mathrm{Fitt}_{\mathbb{Z}_p[G]}(\mathrm{Tor}_{\mathbb{Z}_p}\mathfrak{X}_F) = \text{image in } \mathbb{Z}_p[G] \text{ of } \mathrm{Fitt}_{\Lambda[G]}(\mathfrak{X}_\infty)$$

and

$$\mathrm{Sol}_1(F) = \mathrm{Fitt}_{\mathbb{Z}_p[G]}((\mathcal{U}_\infty/\overline{C}_\infty)_\Gamma) = \text{image in } \mathbb{Z}_p[G] \text{ of } \mathrm{Fitt}_{\Lambda[G]}(\mathcal{U}_\infty/\overline{C}_\infty)$$

The Main Conjecture implies that $\mathcal{U}_\infty/\overline{C}_\infty$ and $\mathfrak{X}_\infty$ have the same characteristic series in Λ, but to compare their $\Lambda[G]$-Fitting ideals, more is needed, some kind of "equivariant Iwasawa theory" which does not exist yet (but there is work in progress, see e.g. [BG], [RW1] ...). Besides, the Fitting ideal of a module is a rather weak invariant, except if this module is of projective dimension at most 1 (see the comments in [Gr3]). For all these reasons, we'll be content to work in the following setting (which was already that of [BB1], [BN], [Gr3]) :

Notations 6.2 *Write $G = P \times \Delta$, where P (resp. Δ) is the p-part (resp. non-p-part) of G.*

For any $\mathbb{Q}_p$-irreducible character ψ of Δ, let e_ψ be the usual idempotent, and for any $\mathbb{Z}_p[G]$ or $\Lambda[G]$-module M, let M_ψ be $e_\psi M$. We intend to compare $MW(F)_\psi$ and $\mathrm{Sol}_1(F)_\psi$ for some non trivial ψ. Let us denote by $\mathbb{Z}_p(\psi)$ (resp. $\Lambda(\psi)$) the rings $e_\psi\mathbb{Z}_p[\Delta] \simeq \mathbb{Z}_p[\chi(\Delta)]$ (resp. $e_\psi\Lambda[\Delta] \simeq \Lambda[\chi(\Delta)]$), where χ is any $\mathbb{Q}_p^{\mathrm{sep}}$-irreducible character of Δ dividing ψ.

We must first show results on projective dimensions, and for that introduce some more appropriate objects and hypotheses :

Definition 6.3

(*i*) *Let $T = S \bigcup \mathrm{Ram}(F/\mathbb{Q})$ (recall that S is the set of primes above p). Slightly abusing language, we'll keep the notations S and T when going up the cyclotomic tower. We define $\mathfrak{X}_\infty^T$ as the Galois group over F_∞ of the maximal abelian pro-p-extension of F_∞ which is unramified outside T.*

(*ii*) *Let Σ be the set of places $v \in T - S$ which split totally in $F(\zeta_p)/F$. A $\mathbb{Q}_p$-irreducible character ψ of Δ will be called "locally non Teichmüller" if either Σ is empty or the restriction of ψ to the decomposition subgroup Δ_v, for any $v \in \Sigma$, differs from the restriction of the Teichmüller character. (Examples of fields for which any character of Δ is locally non Teichmüller may be found in [BB1] and [Gr3]).*

Lemma 6.4

(*i*) *For any $\mathbb{Q}_p$-irreducible character ψ of Δ, $\psi \neq 1$, we have $\mathrm{pd}(\mathfrak{X}_\infty^T)_\psi \leq 1$ over the algebra $\Lambda(\psi)[P]$.*

(*ii*) *For any locally non Teichmüller character $\psi \neq 1$ of Δ, we have $\mathrm{pd}(\mathfrak{X}_\infty)_\psi \leq 1$ over the algebra $\Lambda(\psi)[P]$.*

((*ii*) generalizes corresponding results of [BB1] and [Gr3]).
Proof. By [N1], proposition 1.7, there exists a canonical $\Lambda[G]$-module $\mathcal{Y}_\infty^T$ of projective dimension at most 1, such that $\mathfrak{X}_\infty^T$ and $\mathcal{Y}_\infty^T$ take place in an exact sequence :

$$0 \longrightarrow \mathfrak{X}_\infty^T \longrightarrow \mathcal{Y}_\infty^T \longrightarrow \Lambda[G] \longrightarrow \mathbb{Z}_p \longrightarrow 0$$

By cutting out by any non trivial character ψ of Δ, we get an exact sequence of $\Lambda(\psi)[P]$-modules

$$0 \longrightarrow (\mathfrak{X}_\infty^T)_\psi \longrightarrow (\mathcal{Y}_\infty^T)_\psi \longrightarrow \Lambda(\psi)[P] \longrightarrow 0$$

in which the last two modules have projective dimension less than or equal to 1. Then classical relations between projective dimension in a short exact sequence (see e.g. [N2], proposition 3.3) show immediately the assertion (*i*).

To go back to $(\mathfrak{X}_\infty)_\psi$, recall that Leopoldt's weak conjecture (which is valid for the cyclotomic $\mathbb{Z}_p$ extension) yields an exact sequence

$$0 \longrightarrow \oplus_{w\in\Sigma} M_w \longrightarrow \mathfrak{X}_\infty^T \longrightarrow \mathfrak{X}_\infty^S \longrightarrow 0$$

where M_w is the module obtained by inducing $\mathbb{Z}_p(1)$ from $\mathrm{Gal}(F_\infty/\mathbb{Q})_w$ to $\mathrm{Gal}(F_\infty/\mathbb{Q})$ ([N2],[Wi]). Cutting out by any locally non Teichmüller character ψ of Δ gives the assertion (ii), since the hypothesis "locally non Teichmüller" just means that $(\oplus_{w\in\Sigma} M_w)_\psi = 0$.

□

As $A := \Lambda(\psi)[P]$ is a local ring, lemma 6.4 (i) gives us a short free resolution $0 \longrightarrow A^m \xrightarrow{\phi} A^m \longrightarrow (\mathfrak{X}_\infty^T)_\psi \longrightarrow 0$, and $\mathrm{Fitt}_A(\mathfrak{X}_\infty^T)_\psi = (\det\phi)$. Let us abbreviate $R := \mathrm{Ram(F/\mathbb{Q})}$, denote $\det\phi$ by $H_R = H_{R,\psi} \in A$, and call it "the" *equivariant characteristic series* of the A-module $(\mathfrak{X}_\infty^T)_\psi$. Lemma 6.4 (ii) gives us an analogous exact sequence for $(\mathfrak{X}_\infty)_\psi$ and ψ locally non-trivial, and analogous *equivariant characteristic series*, $H = H_\psi$ of the A-module $(\mathfrak{X}_\infty)_\psi$.

Let $\widetilde{\psi}$ be the (non-irreducible) character of G induced by ψ. For any $\mathbb{Q}_p^{\mathrm{sep}}$-irreducible character χ of G dividing $\widetilde{\psi}$, let us denote again by χ the map obtained by extending χ to $\mathbb{Z}_p[G]$ or $\Lambda[G]$ by linearity.

Proposition 6.5 *Let ψ be a non-trivial $\mathbb{Q}_p$-irreducible character of Δ, then :*

(i) $\mathrm{Fitt}_{\mathbb{Z}_p(\psi)[P]}(\mathfrak{X}_F^T)_\psi = (H_R(0))$.

(ii) *If moreover ψ is locally non Teichmüller, then* $\mathrm{Fitt}_{\mathbb{Z}_p(\psi)[P]}(\mathfrak{X}_F)_\psi = (H(0))$.

(iii) *For any $\mathbb{Q}_p^{sep}$-irreducible character χ of G dividing $\widetilde{\psi}$:*
$\mathrm{Fitt}_{\Lambda(\chi)}((\mathfrak{X}_\infty^T)_\chi) = (\chi(H_R)) = (h_{R,\chi}(T))$, *where $h_{R,\chi(T)}$ is the usual characteristic series of $(\mathfrak{X}_\infty^T)_\chi$ over $\Lambda(\chi)$. Here M_χ denotes the "χ-quotient" of M (see e.g. [Ts],§2).*

(iv) *For any $\mathbb{Q}_p^{\mathrm{sep}}$-irreducible character χ of G :*
$\mathrm{Fitt}_{\Lambda(\chi)}((\mathfrak{X}_\infty)_\chi) = (\chi(H)) = (h_\chi(T))$, *where $h_{\chi(T)}$ is the usual characteristic series of $(\mathfrak{X}_\infty)_\chi$ over $\Lambda(\chi)$.*

Proof. (i) and (ii) are direct consequences of lemma 6.4, (iii) and (iv) are the classical Main Conjecture (see e.g. [Gr1] [Ts]).

□

Recall that by the Main Conjecture, $\chi(H)(0) \overset{p}{\sim} h_\chi(0) \overset{p}{\sim} L_p(\chi,1)$ (see e.g. [Gr1], theorem 3.2 and proposition 3.4).

Let us now deal with $\mathrm{Sol}_1(F) = \mathrm{Fitt}_{\mathbb{Z}_p[G]}((\mathcal{U}_\infty/\overline{C}_\infty)_\Gamma)$. To this end, we introduce some "structural constants" which intervene in the computation of Sinnott's constant "character by character" (see [BN])

Definition 6.6 *For any $\mathbb{Q}_p^{\mathrm{sep}}$-irreducible character χ of G, let F_χ be the field cut out by χ (i.e. the fixed field of $\mathrm{Ker}\,\chi$). For all subfields $M \subset F$ such that $F_\chi \subset M$, define $b_{\chi,M}^F := [F:M]\prod_\ell(1-\chi^{-1}(\ell))$, where the product is taken over all primes ℓ dividing the conductor of M. Let b_χ^F be "the" greatest common divisor of all the $b_{\chi,M}^F$.*

(actually since $O_{F_w}(\chi)$ is a local ring, b_χ^F is "equal" to one of the $b_{\chi,M}^F$ with minimal p-adic valuation). One can easily check that the ideal (b_χ^F) defined here coincides with (b_χ) defined in [BN], définition 1.6.

Lemma 6.7 *Suppose that p is totally split in F. For any $\chi \neq 1$, we have $\chi(\mathrm{Sol}_1(F)) = (b_\chi^F L_p(\chi,1))$ in $\mathbb{Z}_p(\chi)$.*

Proof. If $F_\chi \not\subset M$, then obviously $\chi(\mathrm{sol}_M^F) = 0$. By definition of b_χ^F it will be enough to show that if $F_\chi \subset M$, then $\chi(\mathrm{sol}_M^F) \overset{p}{\sim} b_{\chi,M}^F L_p(\chi,1)$. We first consider the special case $M = F$ and χ has the same conductor, f say, as F. In that case we have $\mathrm{sol}_F^F = \mathrm{sol}_F$ and $b_{\chi,F}^F = 1$. It follows easily that

$$\begin{aligned}
\chi(\mathrm{sol}_F) &= \frac{1}{p} \sum_{g \in \mathrm{Gal}(F/\mathbb{Q})} \log_p(\imath(\varepsilon_F^g))\chi(g^{-1}) \\
&= \frac{1}{p} \sum_{g \in \mathrm{Gal}(F/\mathbb{Q})} \log_p(\imath(N_{\mathbb{Q}(\zeta_f)/F}(1-\zeta_f)^g))\chi(g^{-1}) \\
&= \frac{1}{p} \sum_{g \in \mathrm{Gal}(\mathbb{Q}(\zeta_f/\mathbb{Q}))} \log_p(\imath((1-\zeta_f)^g))\chi(g^{-1}) \\
&= \frac{1}{p} \sum_{1 \leq a \leq f} \log_p(\imath(1-\zeta_f^a))\chi^{-1}(a) \\
&\overset{p}{\sim} L_p(\chi,1),
\end{aligned}$$

(the last p-adic equivalence comes from a classical formula for $L_p(\chi,1)$; see e.g [W], p. 63). Note that an implicit choice of $\imath$ is made there, and that since we are assuming that p is not ramified in F, the quantities $p - \chi(p)$, f_χ, and $\tau(\chi)$ are all p-adic units.

Now $\chi(\widetilde{\mathrm{sol}}_{F_\chi}^F)$ is well defined because taking another lift would only add an element of the (additive) kernel of χ. From the previous special case we deduce the equivalence (without assuming anything on the conductor f_χ) : $\chi(\widetilde{\mathrm{sol}}_{F_\chi}^F) = \chi(\mathrm{sol}_{F_\chi}^{F_\chi}) \overset{p}{\sim} L_p(\chi,1)$, hence $\chi(\mathrm{sol}_{F_\chi}^F) \sim \chi(\widetilde{\mathrm{sol}}_{F_\chi}^F\ \mathrm{Tr}_{F/F_\chi}) \sim [F:F_\chi]\chi(\widetilde{\mathrm{sol}}_{F_\chi}^F) \sim [F:F_\chi]L_p(\chi,1) \sim b_{\chi,F_\chi}^F L_p(\chi,1)$. The remaining cases are treated using the distribution relations satisfied by the cyclotomic numbers (and hence by the elements sol_M^F) and the formal identity $\chi(\mathrm{sol}_M^F) = \frac{1}{[M:F_\chi]}\chi(\mathrm{Tr}_{M/F_\chi}\mathrm{sol}_M^F)$.
□

Corollary 6.8 *$MW(F) \neq \mathrm{Sol}_1(F)$ in general. But in the semi-simple case (i.e. $p \nmid [F:\mathbb{Q}]$), $MW(F) = \mathrm{Sol}_1(F)$.*

Proof. In the semi-simple case, for any $\chi \neq 1$, b_χ^F is a p-adic unit, the preceding calculation about P-cohomology becomes empty. As for the trivial character, the corresponding idempotent is just the norm map, which brings us down to $\mathbb{Q}$, where nothing harmful happens.
□

Theorem 6.9 *Suppose that p is totally split in F. For any non trivial and locally non Teichmüller $\mathbb{Q}_p$-irreducible character ψ of Δ, the following properties are equivalent :*

(i) $(\overline{C}_\infty)_\psi$ *is* $\Lambda(\psi)[P]$*-free*

(ii) $(\mathfrak{X}_\infty)_\psi$ *and* $(\mathcal{U}_\infty/\overline{C}_\infty)_\psi$ *have "the" same equivariant characteristic series in* $\Lambda(\psi)[P]$

(iii) $(KN_F)_\psi = 0$ *and* $(\widetilde{C}''_F)_\psi$ *is* $\mathbb{Z}_p(\psi)[P]$*-free*

They all imply :

(iv) $MW(F)_\psi = \mathrm{Sol}_1(F)_\psi$

(v) $(\widehat{U}'_F/\widetilde{C}''_F)^*_\psi$ *and* $(X'_\infty)_{\Gamma,\psi}$ *have the same Fitting ideal over* $\mathbb{Z}_p(\psi)[P]$, $(.)^*$ *denoting the Pontryagin dual, endowed with the Galois action defined by* $f^\sigma(x) = f(\sigma x)$.

Proof. To show the equivalence between (i) and (ii), let us consider the module $(\mathcal{U}_\infty/\overline{C}_\infty)_\psi \simeq (\mathcal{U}_\infty)_\psi/(\overline{C}_\infty)_\psi$. Since p is totally split in F, for all $v \in S$ the local fields F_v contain no p^{th} power root of unity. Coleman's theory (see e.g. the exact sequence in theorem 4.2 of [Ts]) then shows that $\mathcal{U}_\infty$ is a rank one $\Lambda[G]$-free module, hence $(\mathcal{U}_\infty)_\psi \simeq A := \Lambda(\psi)[P]$. This gives the short resolution $0 \longrightarrow (\overline{C}_\infty)_\psi \longrightarrow A \longrightarrow (\mathcal{U}_\infty/\overline{C}_\infty)_\psi \longrightarrow 0$, which shows the equivalences :
$\mathrm{pd}_R(\mathcal{U}_\infty/\overline{C}_\infty)_\psi \le 1 \iff (\overline{C}_\infty)_\psi$ is R-free $\iff \mathrm{Fitt}_R(\mathcal{U}_\infty/\overline{C}_\infty)_\psi$ is principal.

In the last eventuality, let us denote by J the equivariant characteristic series of $(\mathcal{U}_\infty/\overline{C}_\infty)_\psi$ (as defined before 6.5). To compare J with H (the equivariant characteristic series of $(\mathfrak{X}_\infty)_\psi$), we appeal to still another algebraic lemma of Greither :

Lemma 6.10 ([Gr3], 3.7) *Let $A = \Lambda(\psi)[P]$. If M is an A-torsion module of projective dimension at most 1, such that M/pM is finite, and if J is an element of A such that* $\mathrm{Fitt}_{\Lambda(\chi)}(M_\chi) = (\chi(J))$ *for all* $\mathbb{Q}_p^{\mathrm{sep}}$*-irreducible characters χ of G dividing $\widetilde{\psi}$, then actually* $\mathrm{Fitt}_A(M) = (J)$.

We apply this to $M = (\mathcal{U}_\infty/\overline{C}_\infty)_\psi$. It is known (see [BN], [Ts]) that for all $\chi \mid \widetilde{\psi}$, the $\Lambda(\chi)$-modules $(\mathcal{U}_\infty/\overline{C}_\infty)_\chi$ and $(\mathfrak{X}_\infty)_\chi$ have the same (usual) characteristic series. In particular, their μ-invariants are null. It follows that $\chi(H) = \chi(J)$ for all $\chi \mid \widetilde{\psi}$, hence $(H) = (J)$ by Greither's lemma. Conversely the equality $(H) = \mathrm{Fitt}_{\Lambda(\psi)[P]}(\mathcal{U}_\infty/\overline{C}_\infty)_\psi$ implies the principality of this last ideal. The proof of the equivalence between (i) and (ii) is thus complete. Property (i) implies the triviality of $(KN_F)_\psi$ and then (iii) by Γ-co-descent.

Conversely, assume (iii) and choose a $\mathbb{Z}_p(\psi)[P]$-basis $(\overline{y_i})$ of $(\widetilde{C}''_F)_\psi$. By Nakayama's lemma, this can be lifted to a system of $\Lambda(\psi)[P]$-generators (y_i) of $(\overline{C}''_\infty)_\psi$. Any linear relation $\sum \lambda_i y_i = 0$, with $\lambda_i \in \Lambda(\psi)[P]$, would give, by Γ-co-descent, $\overline{\lambda_i} = 0$ in $\mathbb{Z}_p(\psi)[P]$ for any i, viz. $T (:= \gamma - 1)$ would divide all the coefficients λ_i. As $(\overline{C}_\infty)_\psi$ has no T-torsion, we could then simplify by T in the above linear relation and repeat the process. This shows that the system (y_i) is a $\Lambda(\psi)[P]$-basis.

By the same argument, (ii) implies (iv). Property (ii) implies (v) because of the (ψ)-parts of the exact sequence (7) together with one last algebraic lemma :

Lemma 6.11 *Let* $0 \to M \to N \to N' \to M' \to 0$ *be an exact sequence of* $\mathbb{Z}_p(\psi)[P]$*-modules of finite order. Suppose that* N *and* N' *are of projective dimension at most* 1 *and have the same Fitting ideal over* $\mathbb{Z}_p(\psi)[P]$*. Then* M^* *and* M' *have the same Fitting ideal over* $\mathbb{Z}_p(\psi)[P]$.

Proof. This is [CG], Proposition 6, but note that the correct statement involves M^* (and not M as in [CG]).
□

Acknowledgement : We thank Radan Kučera for pointing out an error in a previous formulation of lemma 5.3.

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