# ANNIHILATION OF REAL CLASSES

*by*

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***Abstract.*** — Let $F$ be a number field, abelian over $\mathbb{Q}$ and let $p$ be a prime unramified in $F$. In this note we prove that Solomon's $\psi_F$ element annihilates the torsion of the Galois group of the maximal abelian $p$-ramified $p$-extension of $F$. This is another unconditional proven reinforcement of All theorem [**All13**] which was conjecture 4.1 of [**Sol92**].

## 1. Introduction

Let $F$ be a number field, abelian over $\mathbb{Q}$, and fix a prime $p \neq 2$. For all $n \in \mathbb{N}$ fix $\zeta_n$ a primitive $n^{\text{th}}$ root of unity in a coherent way, e.g. take $\zeta_n = e^{2i\pi/n}$. Let $f$ be the conductor of $F$, so that $F \subset \mathbb{Q}(\zeta_f)$ and $f$ is minimal with respect to that property. Let $X_F$ denotes the $p$-part of the ideal class group of $F$. The Galois group $\Delta_F = \Delta = \mathrm{Gal}(F/\mathbb{Q})$ acts onto $X_F$ in a natural way, turning it into a $\mathbb{Z}_p[\Delta]$-module. One of the main challenge of algebraic number theory is to understand the structure of this object. As this is a finite module, a starting point would be to describe its annihilator or to find some canonical annihilators. If $F$ is imaginary the Stickelberger element is such canonical element for the minus part of $X_F$. However if $F$ is real this Stickelberger element is a multiple of the absolute trace, hence the annihilation statement is trivial. Assume that $F$ is totally real. Fix once and for all an embedding of $\overline{\mathbb{Q}}$ in the field of complex number $\mathbb{C}$ and another embedding $\imath_v \colon \overline{\mathbb{Q}} \longrightarrow \mathbb{C}_p$ in the Tate field $\mathbb{C}_p$. This embedding uniquely defines a place $v$ above $p$ in $F$. Let $F_v \in \mathbb{C}_p$ be the closure of $F$ and let $\mathcal{O}_v$ be the local valuation ring at $v$. In [**Sol92**] Solomon defined an element $\psi_F \in \mathcal{O}_v[\Delta]$ as follows

$$\psi_F := \frac{1}{p} \sum_{\delta \in \Delta_F} \log_p \imath_v \big( N_{\mathbb{Q}(\zeta_f)/F} (1 - \zeta_f)^{\delta} \big) \delta^{-1},$$

and stated the conjecture :

***Conjecture 1.1*** **([Sol92], conjecture 4.1)**. — *$\psi_F$ annihilates $X_F \otimes \mathcal{O}_v$.*

Let us state a few historical remarks, for which we need some more notations. Let $\mathfrak{X}_F$ be the Galois group of the maximal abelian $p$-ramified $p$-extension of $F$ and let

$t\mathfrak{X}_F \subset \mathfrak{X}_F$ be its $\mathbb{Z}_p$-torsion module. Let also $\mathcal{D}_F$ be the sub-module of $X_F$ generated the places dividing $p$.

1. If $p \nmid |\Delta|$ (so called semi-simple case) then conjecture 1.1 follows from the Main Conjecture of Iwasawa theory (here Mazur-Wiles theorem [**MW84**]) : see remark 4.1 $(ii)$ of [**Sol92**].
2. Theorem 4.1 of [**Sol92**] proves that $\psi_F$ annihilates $\mathcal{D}_F \otimes \mathcal{O}_v$.
3. If $f$ is composite and if $p$ is totally split in $F$, then theorem 5.4 of [**BNQD05**] proves conjecture 1.1.
4. In full generality the theorem 1.1 of [**All13**] is a proven reinforcement of conjecture 1.1.

The point here is to construct global explicit annihilators of all $X_F \otimes \mathcal{O}_v$ inside $\mathbb{Z}_p[\Delta]$ without using $\chi$-eigenspaces, so the semi-simple cases are not relevant to this problem. The main result of this note is

***Theorem 1.2***. — *$\psi_F$ annihilates $t\mathfrak{X}_F \otimes \mathcal{O}_v$.*

As $t\mathfrak{X}_F$ maps surjectively onto $X_F$, this theorem is another reinforcement of conjecture 1.1.

The theorem 1.2 reinforces conjecture 1.1, but it also proves that the element $\psi_F$ is not a real analogue of Stickleberger element, in the sense that $\psi_F$ is more naturally associated to $t\mathfrak{X}_F$ than to $X_F$. The interested reader should consult [**NQDN11**] and the pioneering work [**Sol09**] where a better real analogue of Stickelberger ideal are introduced and studied. The strategy to prove theorem 1.2 is quite simple. One first use Iwasawa theory to consider objects $\mathfrak{X}_\infty$ and $X_\infty$ defined by replacing $F$ with its cyclotomic $\mathbb{Z}_p$-extension $F_\infty$. Then one use Coleman theory to define a global power series $Col(\eta_f) \in \mathcal{O}_v[\Delta][[T]]$ such that $\psi_F$ is recovered from $Col(\eta_f)$ simply by setting $T = 0$. Now by classical Iwasawa's Main Conjecture, which is a statement about $\chi$-parts, all $\chi$ components of $Col(\eta_f)$ annihilates all $\chi$-parts of $\mathfrak{X}_\infty$. But the point here is that $\mathfrak{X}_\infty$ has simultaneously no finite sub-module and trivial $\mu$-invariant, contrary to $X_\infty$ which is conjectured to be finite. Therefore the global elements $Col(\eta_f)$ actually annihilates full $\mathfrak{X}_\infty$ and theorem 1.2 follows by taking co-invariants.

## 2. Setting

Let us recall the analytic and algebraic setting of Iwasawa Main Conjecture.

### 2.1. Galois setting.

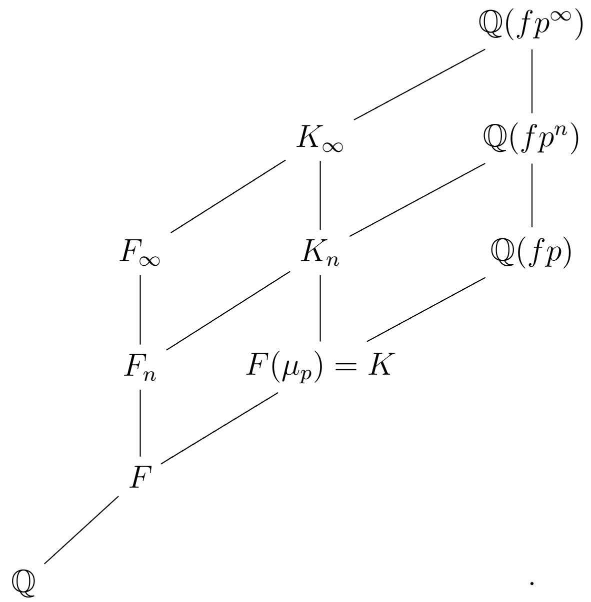
— Fix once and for all a number field $F$, abelian over $\mathbb{Q}$, real, and of conductor $f$. Fix a prime number $p \neq 2$ such that $p \nmid f$. For all integer $n$, the field $F_n$ will be the $n$-th step of the cyclotomic $\mathbb{Z}_p$-extension of $F$, so that $[F_n : F] = p^n$. Also we put $K_n = F(\zeta_{p^n})$ and $\Delta_{K_n} = \mathrm{Gal}(K_n/\mathbb{Q})$, $\Delta_{F_n} = \mathrm{Gal}(F_n/\mathbb{Q})$. As $p \nmid f$, we have $K_n = F_n(\zeta_p)$. Set also $F_\infty = \bigcup_n F_n$ and $K_\infty = \bigcup_n K_n$. By disjoint ramification the relevant extensions are split and we have direct product of various

Galois groups as follows :

$$\Delta_{F_n} \simeq \Delta_F \times \mathrm{Gal}(F_n/F) \simeq \Delta_F \times \mathbb{Z}/p^n\mathbb{Z};$$

$$\Delta_{K_n} \simeq \Delta_F \times \mathrm{Gal}(K/F) \times \mathrm{Gal}(K_n/K) \simeq \Delta_F \times \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z}.$$

The whole Galois setting is summarized as follows




For any finite abelian group $G$ and any valuation ring $\mathcal{O}$ of a finite extension of $\mathbb{Q}_p$, we will denote by $\Lambda_{\mathcal{O}}[G]$ the Iwasawa algebra

$$\Lambda_{\mathcal{O}}[G] = (\varprojlim \mathcal{O}[\mathbb{Z}/p^n\mathbb{Z}])[G] \simeq \mathcal{O}[G][[T]].$$

If $\mathcal{O} = \mathbb{Z}_p$ we simply note $\Lambda[G] = \Lambda_{\mathcal{O}}[G]$. Moreover we will note $\omega$ the *Teichmüller* character defined by the embedding $\imath_v$. Precisely we have for all $a$ in $\mathbb{Z}$ either $p \mid a$ and $\omega(a) = 0$ either $(a,p) = 1$ and $\omega(a)$ is the unique $(p-1)^{\text{th}}$ root of unity (simultaneously in $\mathbb{C}$ and $\mathbb{C}_p$) such that $v(\omega(a) - a) > 0$. Identifying $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ with $(\mathbb{Z}/p\mathbb{Z})^\times$, this character describes the action of Galois on the $p^{\text{th}}$-power root of unity as for all $g$ and all such root of unity $\zeta$ we have $g(\zeta) = \zeta^{\omega(g)}$, the right hand side being well defined because $\omega(g) \in \mathbb{Z}_p$. We also will denote by $\kappa\colon \mathrm{Gal}(\mathbb{Q}(fp^\infty)/\mathbb{Q}(f)) \xrightarrow{\sim} \mathbb{Z}_p^\times$ the canonical isomorphism (abusively called cyclotomic character). For all $a$ in $\mathbb{Z}_p^\times$ we will denote $\sigma_a$ the element of $Gal\big(\mathbb{Q}(\zeta_{fp^\infty}|\mathbb{Q})\big)$ such that $\kappa(\sigma_a) = a$.

**2.2. Analytic side.** — The $p$-adic $L$ function is defined by interpolating the values of the complex $L$ functions at negative integers. To be more precise let us recall a fex classical notations. Let $\chi$ be a Dirichlet character, assumed to be *primitive* modulo $f$. Via the Artin map the character $\chi$ may be seen as a Galois Character, actually a character of $Gal(\mathbb{Q}(\zeta_f)|\mathbb{Q})$. Let $\mathbb{Q}(\chi) \subset \overline{\mathbb{Q}}$ be the field generated over $\mathbb{Q}$ by the values of $\chi$. The $\chi$-twisted Bernoulli numbers $B_{n,\chi} \in \mathbb{Q}(\chi)$ are defined by the power series expansion

$$\sum_{a=1}^{f} \frac{\chi(a)se^{as}}{e^{fs}-1} = \sum_{n=0}^{\infty} B_{n,\chi}\frac{s^n}{n!}.$$

The complex $L$ function $L(s,\chi)$ takes algebraic values at negative integers. Indeed according to [**Was97**] theorem[(1)] one has :

$$L(1-n,\chi) = -\frac{B_{n,\chi}}{n}.$$

Now by making use of the fixed embedding $\imath_v$ it is possible to interpolate these values with a $\mathbb{C}_p$ valued function.

***Theorem 2.1* (cf. [Was97] theorem 5.11)**. — *Exists a unique meromorphic function $L_p(.,\chi)$ (which is holomorphic if $f \neq 1$) defined on the open ball $\{s \in \mathbb{C}_p| \ \ |s| < pp^{-1/p-1}\}$ and such that for all $n \geq 1$ :*

$$L_p(1-n,\chi) = -(1-\chi\omega^{-n}(p)p^{n-1})\frac{B_{n,\chi\omega^{-n}}}{n} = (1-\chi\omega^{-n}(p)p^{n-1})L(1-n,\chi\omega^{-n}).$$

***Remark 2.2***. — If $f \neq 1$, then one has the Leopoldt formula (see [**Was97**] theorem 5.18)

$$L_p(1,\chi) = -\left(1-\frac{\chi(p)}{p}\right)\frac{\tau(\chi)}{f}\sum_{a=1}^{f}\chi(a)^{-1}\log_p \imath_v(1-\zeta_f^a),$$

where $\tau = \tau(\chi) = \sum_{a=1}^{a=f}\chi(a)\zeta_f^a$ is the Gauß sum associated to $\chi$. Therefore the Solomon's element $\psi_F$ is obviously related to these values at 1.

These $p$-adic $L$ functions are uniquely determined by this interpolation properties. Their original definition using Mahler development of partial zeta functions does not fully reveal their real nature. Later on Iwasawa discovered a far more concrete definition by using a sequence of Stickelberger elements to obtain a power series development of these functions. These power series are elements of $\mathbb{Z}_p(\chi)[[T]]$.

***Definition 2.3***. — The Stickelberger element $\xi_{\mathbb{Q}_{\zeta_{fp^n}}}$ related to $\mathbb{Q}(\zeta_{fp^n})$ is defined by the formula

$$\xi_{\mathbb{Q}(\zeta_{fp^n})} = \frac{1}{fp^n}\sum_{(a,fp)=1;a=1}^{a=fp^n-1}\left(\frac{fp^n}{2}-a\right)\sigma_a^{-1} \in \mathbb{Q}[\mathrm{Gal}(\mathbb{Q}(\zeta_{fp^n})/\mathbb{Q})].$$

By definition the Stickelberger element related to $K_n$ is the element deduced from $\xi_{\mathbb{Q}(\zeta_{fp^n})}$ by restriction from $\mathrm{Gal}(\mathbb{Q}(\zeta_{fp^n})/\mathbb{Q})$ to $\Delta_{K_n}$.

$$\xi_{K_n} = \frac{1}{fp^n}\sum_{(a,fp)=1;a=1}^{a=fp^n-1}\left(\frac{fp^n}{2}-a\right)(\sigma_a^{-1})_{|K_n} \in \mathbb{Q}[\Delta_{K_n}].$$

One can then show that

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1. Let $\vartheta_{\mathbb{Q}(\zeta_{fp^n})} = \left(1-(1+fp)(\kappa^{-1}(1+fp))^{-1}\right)\xi_{\mathbb{Q}(\zeta_{fp^n})}$. Then one has
$$\vartheta_{\mathbb{Q}(\zeta_{fp^n})} \in \mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}(\zeta_{fp^n})/\mathbb{Q})],$$
so that its restriction $\vartheta_{K_n}$ to $\mathbb{Q}[\Delta_{K_n}]$ is also a $p$-adic integral element actually an element of $\mathbb{Z}_p[\Delta_{K_n}]$
2. The sequences $(\xi_{K_n})_{n\in\mathbb{N}}$ and therefore $(\vartheta_{K_n})_{n\in\mathbb{N}}$ are coherent with respect to restrictions maps along the towers $\mathbb{Q}(fp^\infty)/\mathbb{Q}(f)$ and $K_\infty/K$.

As the $\vartheta_{K_n}$ are coherent along $K_\infty/K$ we may consider the limit $\vartheta_{K_\infty} \in \Lambda(\Delta_K)$. Let $\chi^* = \chi^{-1}\omega$ be the mirror character of $\chi$. Then $\chi^*$ is a Dirichlet character of $\Delta_K$. As all other character of $\Delta_K$ our $\chi^*$ extends linearly to a ring homomorphism $\chi^*\colon \Lambda(\Delta_K) \longrightarrow \Lambda(\chi)$, where $\Lambda(\chi)$ is just the ring obtained by adjoining to $\Lambda$ the values of $\chi$. Let us fix the topological generator $\gamma = \kappa^{-1}(1+fp)$ of $\Gamma = \mathrm{Gal}(K_\infty/K)$. Then sending $\gamma$ to $1+T$ uniquely define the Serre's isomorphism $\mathcal{S}\colon \Lambda(\chi) \simeq \mathbb{Z}_p(\chi)[[T]]$.

***Definition 2.4.*** — We may now define the three relevant Iwasawa's power series attached to our Dirichlet character $\chi$ :

1. $g(T,\chi) = \mathcal{S}(\chi^*(\vartheta_{K_\infty})) \in \mathbb{Z}_p(\chi)[[T]]$;
2. $h(T,\chi) = 1 - \dfrac{1+pf}{1+T} \in \mathbb{Z}_p[[T]]$;
3. $f(T,\chi) = \dfrac{g(T,\chi)}{h(T,\chi)} \in \mathbb{Z}_p(\chi)[[T]]$.

Iwasawa in [**Iwa69**] discovered that these series would provide another definition of the functions $L_p(s,\chi)$. This definition turned out to be far more enlightening than the original one.

***Theorem 2.5*** **([Iwa69])**. — *Let $\chi$ be an even character of $\Delta_K$. Then for all $s \in \mathbb{C}_p$ such that $|s| < pp^{-1/(p-1)}$ (and $s \neq 1$ if $\chi$ is trivial) we have*
$$L_p(s,\chi) = f((1+fp)^s - 1, \chi).$$

**2.3. Algebraic side.** — For all number field $F$ recall that

– $X_F$ is the $p$-part of the class group of $F$.
– $\mathfrak{X}_k$ is the Galois group of the maximal abelian $p$-ramified $p$-extension of $F$.

We $X_\infty = \varprojlim X_{F_n}$, $\mathfrak{X}_\infty = \varprojlim \mathfrak{X}_{F_n}$ (with respect to norms and restriction maps). These are finitely generated $\Lambda(\Delta_F)$-modules and are linked together by the exact sequence
$$0 \to \overline{E}_\infty \to \mathcal{U}_\infty \to \mathfrak{X}_\infty \to X_\infty \to 0$$
where

– $\overline{E}_\infty$ is the direct limit of the $\overline{E}_n = E_n \otimes \mathbb{Z}_p$ where the $E_n$ are the global units of the $F_n$ .
– $\mathcal{U}_\infty$ is the direct limit of the semi-local units of the $F_n$.

By the weak Leopoldt conjecture (here a theorem) the $\Lambda$-module $\mathfrak{X}_\infty$ is torsion. In particular, for all character $\chi$ of $\Delta_F$, the $\chi$-part $\mathfrak{X}_\infty^\chi$ is a finitely generated torsion $\mathbb{Z}_p(\chi)[[T]]$-module. We therefore may consider its characteristic series that we will denote $c.s.(\mathfrak{X}_\infty^\chi)$.

***Remark 2.6***. — It is a well known theorem of Iwasawa (see [**Iwa59**]) that $\mathfrak{X}_\infty^\chi$ has no non-trivial finite $\Lambda$-submodule. In particular it is annihilated by its characteristic series.

**2.4. Main conjecture.** — The main conjecture of Iwasawa relevant to this context may be stated as follows :

***Theorem 2.7*** **(Mazur-Wiles)**. — *For all character $\chi$ of $\Delta_F$ we have*

$$s.c.(\mathfrak{X}_\infty^\chi)(T) = f^\sharp(T,\chi)$$

*where, by definition, the Iwasawa's involution $\sharp$ is given by*

$$f^\sharp(T,\chi) = f\left(\frac{1+fp}{1+T} - 1, \chi\right).$$

The mirror Stickelberger series $f^\sharp(T,\chi)$ plays an important role here. Another (equivalent) statement of the main conjecture is the following equality of characteristic series :

$$s.c.(\mathfrak{X}_\infty^\psi)(T) = s.c.(\mathcal{U}_\infty/\mathcal{C}_\infty)^\psi(T),$$

where $\mathcal{C}_\infty$ stands by definition for the direct limit of circular units of the $F_n$.

## 3. Solomon's element

Recall that for an abelian totally real field $F$ such that $p$ is unramified in $F$ the Solomon's element is defined by the formula

$$\psi_F := \frac{1}{p}\sum_{\delta\in\Delta_F} \log_p\left(\imath_v(N_{\mathbb{Q}(\zeta_f)/F}(1-\zeta_f)^\delta)\right)\ \delta^{-1}.$$

Our goal in that section is to use Coleman theory to relate this $\psi_F$ to the constant term of a power series that annihilates $\mathfrak{X}_\infty$.

**3.1. Coleman's theory.** — This theory is essentially a *local* one and it stands for any unramified extension of $\mathbb{Q}_p$. Let $\mathcal{O}_v[[X]]$ be the ring of formal power series in one indeterminate $X$ with coefficient in $\mathcal{O}_v$. It is important here not to mix the Coleman $X$ with the Iwasawa's $T$. Let $\mathcal{O}_v((X))$ be the field of fraction of $\mathcal{O}_v[[X]]$. We will also denote $\Gamma^\times = Gal(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p)$; this notation is coherent with the previous notation $\Gamma = \mathrm{Gal}(K_\infty/K)$.

The rings $\mathcal{O}_v[[X]]$ and $\mathcal{O}_v((X))$ are endowed with :

1. the action of a Frobenius $\varphi$ such that $f^\varphi(X) = f((1+X)^p - 1)$.
2. a norm operator $\mathcal{N}$ uniquely determined by $\mathcal{N}(f^\varphi)(X) = \prod_{\xi\in\mu_p} f(\xi(1+X)-1)$.

3. an action of $\mathbb{Z}_p[[\Gamma^\times]]$ which extends by linearity and continuity the action of $\Gamma^\times$ defined by $\gamma.(1+X) = (1+X)^{\kappa(\gamma)}$ for all $\gamma \in \Gamma^\times$ (recall that $\kappa$ is the cyclotomic character).

***Theorem 3.1* (Coleman).** — *Let $(\alpha_n) = \alpha \in \varprojlim F_v(\zeta_{p^n})^\times$ be a norm coherent sequence of local numbers. Then there exists a unique power series $f_\alpha \in \mathcal{O}_v((X))^\times$ such that*

$$f_\alpha(\zeta_{p^{n+1}} - 1) = \alpha_n \quad \text{and} \quad \mathcal{N} f_\alpha = f_\alpha^\varphi.$$

*Let $Col\colon \varprojlim F_v(\zeta_{p^n})^\times \longrightarrow \mathcal{O}_v((X))$ be the map defined by*

$$Col(\alpha) = (1 - \frac{\varphi}{p}) \log_p f_\alpha(X) = \frac{1}{p} \log_p \frac{\big(f_\alpha(X)\big)^p}{f_\alpha^\varphi(X)}.$$

*Then $Col$ induces an exact sequence*

$$1 \longrightarrow \mathbb{Z}_p(1) \longrightarrow U^1_\infty \xrightarrow{Col} \mathcal{R} \xrightarrow{\epsilon} \mathbb{Z}_p(1) \to 0.$$

*where*

- *$U^1_\infty$ is the direct limit of the principal units of $F_v(\zeta_{p^n})$;*
- *$\mathcal{R}$ is a rank one free $\mathcal{O}_v[[\Gamma^\times]]$-module generated by $(1+X)$;*
- *$\mathbb{Z}_p(1)$ is the direct limit of the $(\zeta_{p^n})$;*
- *$\epsilon$ is the map $f \mapsto Df_{X=0}$ where $D = (1+X)\frac{d}{dX}$.*

It is useful to consider another related exact sequence that is obtained by using the *Mellin transform* $\mathcal{L}$ of $Col$ with values in $\mathcal{O}_v[[\Gamma^\times]]$ instead of $\mathcal{R}$. Let us briefly restate this definition : We saw that $\mathcal{R}$ is a rank one free $\mathcal{O}_v[[\Gamma^\times]]$ generated by $(1+X)$. Hence all element $g \in \mathcal{R}$ may uniquely be written as $g(X) = \hat{g}.(1+X)$ with $\hat{g} \in \mathbb{Z}_p[[\Gamma^\times]]$. The element $\hat{g}$ (also denoted by $Mel(g)$) is called the Mellin transform of $g$ and we define $\mathcal{L} = Mel \circ Col$.

*3.1.1. Formulas.* — We need to give a dictionary of formulas between $\mathcal{R}$ and $\mathcal{O}_v[[\Gamma^\times]]$. For all $\hat{g} = \sum a_n \gamma^n \in \mathcal{O}_v[[\Gamma^\times]]$ and all characters $\psi$ of $\Gamma^\times$ we note $\psi(\hat{g}) = \sum a_n \psi(\gamma)^n \in \mathbb{Z}_p(\psi)$. Recall that

$$\sum a_n \gamma^n \ . \ (1+X) = \sum a_n (1+X)^{\kappa^n(\gamma)}$$

where $\kappa$ is the cyclotomic character. It follows that for all $g \in \mathcal{R}, i \in \mathbb{Z}$

- $g(0) = \mathbb{1}\big(\hat{g}\big)$ (where $\mathbb{1}$ stands for the trivial character).
- $D^i g = Tw^i\big(\hat{g}\big)$, where if $\hat{g} = \sum a_n \gamma^n$ then $Tw^i\big(\hat{g}\big) = \sum a_n (\kappa^i(\gamma)\gamma)^n$

One has in particular $\big(D^i g\big)_{X=0} = \mathbb{1}\big(Tw^i(\hat{g})\big) = \kappa^i(\hat{g})$. With all these formulas and notation the twin sequence of the exact sequence of Coleman is now

$$1 \longrightarrow \mathbb{Z}_p(1) \longrightarrow U^1_\infty \xrightarrow{\mathcal{L}} \mathcal{O}_v[[\Gamma^\times]] \xrightarrow{\kappa} \mathbb{Z}_p(1) \to 0. \tag{1}$$

*3.1.2. Induction.* — Let $L$ be the splitting field at $p$ of $F$, so that $p$ is totally split in $L/\mathbb{Q}$ and that primes above $p$ remains inert in $F/L$. In order to recover global information for our field $F$ from the local information collected in $F_v$ one first need to induce along $\mathrm{Gal}(L/\mathbb{Q})$ and consider first the "semi-local" situation. We have $\mathrm{Gal}(F_v/\mathbb{Q}_p) \simeq \mathrm{Gal}(F/L)$. Let us abbreviate $\Delta_L = \mathrm{Gal}(L/\mathbb{Q})$. We apply the (exact) functor $- \otimes_{\mathbb{Z}_p[\mathrm{Gal}(F/L)]} \mathbb{Z}_p[\Delta_F]$ to the sequence (1). We get (see also[**Tsu99**]) :

$$0 \longrightarrow \mathbb{Z}_p[\Delta_L](1) \longrightarrow \mathcal{U}_\infty \longrightarrow \widehat{\mathcal{O}}_F[[\Gamma^\times]] \longrightarrow \mathbb{Z}_p[\Delta_L](1) \longrightarrow 0, \tag{2}$$

where $\widehat{\mathcal{O}}_F := \mathcal{O}_F \otimes \mathbb{Z}_p \simeq \mathbb{Z}_p[\Delta_F] \simeq \mathcal{O}_v \otimes_{\mathbb{Z}_p[\mathrm{Gal}(F/L)]} \mathbb{Z}_p[\Delta_F] \simeq \mathcal{O}_v[\Delta_L]$ because $\mathcal{O}_v$ is Galois free (recall that $p$ is unramified in $F$).

**3.2. cyclotomic units and Solomon's element.** — Consider the cyclotomic elements $\eta_{F_n} := N_{\mathbb{Q}(\zeta_{fp^n})/F_n}(1-\zeta_{fp^n})$. These numbers are cyclotomic units of $F_n$ (recall that $f \neq 1$) and they form a norm coherent sequence (this follows from the well known distribution relation satisfied by the numbers $1-\zeta_n$). This allows us to define :

***Definition 3.2.*** — Soit

$$\eta_{F,\infty} = \Big(N_{\mathbb{Q}(\zeta_{fp^n})/F_n}(1-\zeta_{fp^n})\Big)_{n\in\mathbb{N}}.$$

This is an element of $\varprojlim \mathcal{O}_{F_n}^\times$ that may be seen as an element of $\mathcal{U}_\infty$ using the diagonal embedding.

We can now state and prove the key result of this note :

***Theorem 3.3.*** — *The Solomon's element $\psi_F$ is nothing more than*

$$\psi_F = \frac{1}{p-1}\mathbb{1}\left(\mathcal{L}(\eta_{F,\infty})\right).$$

*Proof.* — By the formulas given in 3.1.1 we have $\mathbb{1}\big(\mathcal{L}(\eta_F)\big) = Col\big(\eta_{F,\infty}\big)(0)$, so that the theorem is equivalent to the identity

$$(p-1)\psi_F = Col\big(\eta_{F,\infty}\big)(0).$$

We shall check this equality. First remark that the set of $p$-places of $F$ is in one to one correspondence with $\Delta_L$ via the map

$$\begin{array}{ccc} \Delta_L & \leftrightarrow & \{v, \mid v \text{ is above } p\} \\ \delta & \mapsto & \delta^{-1}v \end{array}$$

Hence for all $\delta \in \Delta_L$, the $\delta$-component of $\eta_{F,\infty}$ (seen as an element of $\mathcal{U}_\infty$) is

$$\imath_v(N_{\mathbb{Q}(\zeta_{fp^n})/F_n}(1-\zeta_f^{\delta^{-1}}\zeta_{p^n}))_n .$$

Therefore we locally have (at the corresponding place $\delta^{-1}v$) that the associated Coleman series is

$$\imath_v\Big(\prod_{\sigma\in Gal(\mathbb{Q}(\zeta_f)/F)} 1-\zeta_f^{\sigma\delta^{-1}}(1+X)\Big).$$

Therefore, and still at the place $\delta^{-1}v$, we obtain

$$\begin{aligned} Col(\eta_{f,\infty}) &= \left(1-\frac{\varphi}{p}\right) f_{\eta_{f,\infty}}(X) \\ &= \frac{1}{p}\log_p \frac{\left(\prod 1-\zeta_f^{\sigma\delta^{-1}}(1+X)\right)^p}{\prod\left(1-\zeta_f^{\sigma\delta^{-1}}(1+X)^p\right)} \end{aligned}$$

It follows that specializing at $X=0$ we find that the $\delta$-component of $Col(\eta_{f,\infty})(0)$ is

$$\frac{p-1}{p}\log_p \iota_v N_{\mathbb{Q}(\zeta_f)/F}(1-\zeta_f{}^{\delta^{-1}})$$

which is exactly $(p-1)$ times the $\delta$-component of $\psi_F$. □

We now prove that the power series related to $\eta_{f,\infty}$ annihilates $\mathfrak{X}_\infty$.

***Lemma 3.4.*** — *The element $\mathcal{L}(\eta_{F,\infty})$ annihilates $\mathfrak{X}_\infty\otimes\mathcal{O}_v$.*

*Proof.* — By the main conjecture, for all character $\chi$ of $\Delta_F$, the characteristic series of $\mathfrak{X}_\infty^\chi$ (which annihilates $\mathfrak{X}_\infty^\chi$) is the same as the one of $\left(\mathcal{U}_\infty/\mathcal{C}_\infty\right)^\chi$. But for all character $\chi$ of $\Delta_F$ the series $\chi(\mathcal{L}(\eta_F)$ is a multiple of the characteristic series of $\left(\mathcal{U}_\infty/\mathcal{C}_\infty\right)^\chi$. This proves that $\chi(\mathcal{L}(\eta_F))$ annihilates $\mathfrak{X}_\infty^\chi$ for all $\chi$. But the whole $\mathfrak{X}_\infty$ has no non-trivial $p$-torsion (Ferrero-Washington and Iwasawa for the finite $\Lambda$-module, [**FW79, Iwa59**]). The result follows. □

Now our main result is

***Theorem 3.5.*** — *The Solomon elements $\psi_F$ annihilates the $\mathbb{Z}_p$-torsion $t\mathfrak{X}_F$ of $\mathfrak{X}_F$.*

*Proof.* — Putting together the theorem 3.3 and the lemma 3.4 and Iwasawa co-invariant we see that $(p-1)\psi_F$ annihilates $(\mathfrak{X}_\infty)_\Gamma$. But these Iwsawa co-invariant are known to be canonically isomorphic to $t\mathfrak{X}_F$. □

***Remark 3.6.*** — It is an interesting but technically difficult question to precisely relate the global integral Coleman power series $\mathcal{L}(\eta_{F,\infty})$ with the global integral power series $\vartheta_{K_\infty}\in\Lambda(\Delta_K)$ and also to relate both to the Deligne-Ribet pseudo-measure. All these objects seems to be various avatar of the $p$-adic $L$ function but there is some discrepancy between them. We hope to be able to say more in a future work.

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