ANNIHILATION OF REAL CLASSES
by
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Abstract. — Let $F$ be a number field, abelian over $\mathbb{Q}$ and let $p$ be a prime unramified in $F$. In this note we prove that Solomon’s $\psi_F$ element annihilates the torsion of the Galois group of the maximal abelian $p$-ramified $p$-extension of $F$. This is another unconditional proven reinforcement of all theorem [All13] which was conjecture 4.1 of [Sol92].

1. Introduction

Let $F$ be a number field, abelian over $\mathbb{Q}$, and fix a prime $p \neq 2$. For all $n \in \mathbb{N}$ fix $\zeta_n$ a primitive $n^{th}$ root of unity in a coherent way, e.g. take $\zeta_n = e^{2\pi i/n}$. Let $f$ be the conductor of $F$, so that $F \subset \mathbb{Q}(\zeta_f)$ and $f$ is minimal with respect to that property. Let $X_F$ denotes the $p$-part of the ideal class group of $F$. The Galois group $\Delta_F = \Delta = \text{Gal}(F/\mathbb{Q})$ acts onto $X_F$ in a natural way, turning it into a $\mathbb{Z}_p[\Delta]$-module. One of the main challenges of algebraic number theory is to understand the structure of this object. As this is a finite module, a starting point would be to describe its annihilator or to find some canonical annihilators. If $F$ is imaginary the Stickelberger element is such canonical element for the minus part of $X_F$. However, if $F$ is real this Stickelberger element is a multiple of the absolute trace, hence the annihilation statement is trivial. Assume that $F$ is totally real. Fix once and for all an embedding of $\overline{\mathbb{Q}}$ in the field of complex number $\mathbb{C}$ and another embedding $\iota_v : \mathbb{Q} \rightarrow \mathbb{C}$ in the Tate field $\mathbb{C}_p$. This embedding uniquely defines a place $v$ above $p$ in $F$. Let $F_v \subset \mathbb{C}_p$ be the closure of $F$ and let $\mathcal{O}_v$ be the local valuation ring at $v$. In [Sol92] Solomon defined an element $\psi_F \in \mathcal{O}_v[\Delta]$ as follows

$$\psi_F := \frac{1}{p} \sum_{\delta \in \Delta_F} \log_p \iota_v \left( N_{\mathbb{Q}(\zeta_f)/F}(1 - \zeta_f)^{\delta} \right) \delta^{-1},$$

and stated the conjecture:

Conjecture 1.1 ([Sol92], conjecture 4.1). — $\psi_F$ annihilates $X_F \otimes \mathcal{O}_v$.

Let us state a few historical remarks, for which we need some more notations. Let $\mathfrak{X}_F$ be the Galois group of the maximal abelian $p$-ramified $p$-extension of $F$ and let
$tX_F \subset X_F$ be its $\mathbb{Z}_p$-torsion module. Let also $D_F$ be the sub-module of $X_F$ generated the places dividing $p$.

1. If $p \nmid |\Delta|$ (so called semi-simple case) then conjecture 1.1 follows from the Main Conjecture of Iwasawa theory (here Mazur-Wiles theorem [MW84]) : see remark 4.1 (ii) of [Sol92].
2. Theorem 4.1 of [Sol92] proves that $\psi_F$ annihilates $D_F \otimes \mathcal{O}_v$.
3. If $f$ is composite and if $p$ is totally split in $F$, then theorem 5.4 of [BNQD05] proves conjecture 1.1.
4. In full generality the theorem 1.1 of [All13] is a proven reinforcement of conjecture 1.1.

The point here is to construct global explicit annihilators of all $X_F \otimes \mathcal{O}_v$ inside $\mathbb{Z}_p[\Delta]$ without using $\chi$-eigenspaces, so the semi-simple cases are not relevant to this problem. The main result of this note is

**Theorem 1.2.** — $\psi_F$ annihilates $tX_F \otimes \mathcal{O}_v$.

As $tX_F$ maps surjectively onto $X_F$, this theorem is another reinforcement of conjecture 1.1.

The theorem 1.2 reinforces conjecture 1.1, but it also proves that the element $\psi_F$ is not a real analogue of Stickleberger element, in the sense that $\psi_F$ is more naturally associated to $tX_F$ than to $X_F$. The interested reader should consult [NQDN11] and the pioneering work [Sol09] where a better real analogue of Stickelberger ideal are introduced and studied. The strategy to prove theorem 1.2 is quite simple. One first use Iwasawa theory to consider objects $\mathcal{X}_\infty$ and $X_\infty$ defined by replacing $F$ with its cyclotomic $\mathbb{Z}_p$-extension $F_\infty$. Then one use Coleman theory to define a global power series $Col(\eta_f) \in \mathcal{O}_v[\Delta][[T]]$ such that $\psi_F$ is recovered from $Col(\eta_f)$ simply by setting $T = 0$. Now by classical Iwasawa’s Main Conjecture, which is a statement about $\chi$-parts, all $\chi$ components of $Col(\eta_f)$ annihilates all $\chi$-parts of $X_\infty$. But the point here is that $X_\infty$ has simultaneously no finite sub-module and trivial $\mu$-invariant, contrary to $X_\infty$ which is conjectured to be finite. Therefore the global elements $Col(\eta_f)$ actually annihilates full $X_\infty$ and theorem 1.2 follows by taking co-invariants.

**2. Setting**

Let us recall the analytic and algebraic setting of Iwasawa Main Conjecture.

2.1. **Galois setting.** — Fix once and for all a number field $F$, abelian over $\mathbb{Q}$, real, and of conductor $f$. Fix a prime number $p \neq 2$ such that $p \nmid f$. For all integer $n$, the field $F_n$ will be the $n$-th step of the cyclotomic $\mathbb{Z}_p$-extension of $F$, so that $[F_n : F] = p^n$. Also we put $K_n = F(\zeta_{p^n})$ and $\Delta_{K_n} = \text{Gal}(K_n/\mathbb{Q})$, $\Delta_{F_n} = \text{Gal}(F_n/\mathbb{Q})$. As $p \nmid f$, we have $K_n = F_n(\zeta_p)$. Set also $F_\infty = \bigcup_n F_n$ and $K_\infty = \bigcup_n K_n$. By disjoint ramification the relevant extensions are split and we have direct product of various
Galois groups as follows:

\[ \Delta_{F_n} \simeq \Delta_F \times \text{Gal}(F_n/F) \simeq \Delta_F \times \mathbb{Z}/p^n\mathbb{Z}; \]

\[ \Delta_{K_n} \simeq \Delta_F \times \text{Gal}(K/F) \times \text{Gal}(K_n/K) \simeq \Delta_F \times \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z}. \]

The whole Galois setting is summarized as follows:

\[ \mathbb{Q}(fp) \leftarrow \cdots \leftarrow K \leftarrow \cdots \leftarrow \mathbb{Q}(fp^n) \]

\[ F_\infty \leftarrow K_n \leftarrow F \]

\[ F_\infty \leftarrow F(\mu_p) = K \leftarrow F \]

If \( \mathcal{O} = \mathbb{Z}_p \) we simply note \( \Lambda[G] = \Lambda_{\mathcal{O}}[G] \). Moreover we will note \( \omega \) the Teichmüller character defined by the embedding \( \iota_v \). Precisely we have for all \( a \) in \( \mathbb{Z}_p \) either \( p \mid a \) and \( \omega(a) = 0 \) either \( (a,p) = 1 \) and \( \omega(a) \) is the unique \((p-1)^{th}\) root of unity (simultaneously in \( \mathbb{C} \) and \( \mathbb{C}_p \)) such that \( v(\omega(a) - a) > 0 \). Identifying \( \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \) with \((\mathbb{Z}/p\mathbb{Z})^\times\), this character describes the action of Galois on the \( p \)th-power root of unity as for all \( g \) and all such root of unity \( \zeta \) we have \( g(\zeta) = \zeta^{\omega(g)} \), the right hand side being well defined because \( \omega(g) \in \mathbb{Z}_p \). We also will denote by \( \kappa: \text{Gal}(\mathbb{Q}(fp^\infty)/\mathbb{Q}(f)) \xrightarrow{\sim} \mathbb{Z}_p^\times \) the canonical isomorphism (abusively called cyclotomic character). For all \( a \) in \( \mathbb{Z}_p^\times \) we will denote \( \sigma_a \) the element of \( \text{Gal}(\mathbb{Q}(\zeta_{fp^\infty}|\mathbb{Q})) \) such that \( \kappa(\sigma_a) = a \).

### 2.2. Analytic side

The \( p \)-adic \( L \) function is defined by interpolating the values of the complex \( L \) functions at negative integers. To be more precise let us recall a few classical notations. Let \( \chi \) be a Dirichlet character, assumed to be primitive modulo \( f \). Via the Artin map the character \( \chi \) may be seen as a Galois Character, actually a character of \( \text{Gal}(\mathbb{Q}(\zeta_f)|\mathbb{Q}) \). Let \( \mathbb{Q}(\chi) \subset \overline{\mathbb{Q}} \) be the field generated over \( \mathbb{Q} \) by the values of \( \chi \). The \( \chi \)-twisted Bernoulli numbers \( B_{n,\chi} \in \mathbb{Q}(\chi) \) are defined by
the power series expansion
\[
\sum_{a=1}^{f} \frac{\chi(a) s e^{as}}{e^{as} - 1} = \sum_{n=0}^{\infty} B_{n, \chi} \frac{s^n}{n!}.
\]

The complex \(L\) function \(L(s, \chi)\) takes algebraic values at negative integers. Indeed according to [Was97] theorem\(^{(1)}\) one has:
\[
L(1-n, \chi) = -\frac{B_{n, \chi} n}{n}.
\]

Now by making use of the fixed embedding \(\bar{v}\) it is possible to interpolate these values with a \(\mathbb{C}_p\) valued function.

**Theorem 2.1** (cf. [Was97] theorem 5.11). — Exists a unique meromorphic function \(L_p(\cdot, \chi)\) (which is holomorphic if \(f \neq 1\)) defined on the open ball \(\{ s \in \mathbb{C}_p | |s| < pp^{-1}/p-1 \}\) and such that for all \(n \geq 1\):
\[
L_p(1-n, \chi) = -(1 - \chi \omega^{-n}(p)p^{n-1}) \frac{B_{n, \chi \omega^{-n}}}{n} = (1 - \chi \omega^{-n}(p)p^{n-1}) L(1-n, \chi \omega^{-n}).
\]

**Remark 2.2.** — If \(f \neq 1\), then one has the Leopoldt formula (see [Was97] theorem 5.18)
\[
L_p(1, \chi) = -\left(1 - \frac{\chi(p)}{p}\right) \frac{\tau(\chi)}{f} \sum_{a=1}^{f} \chi(a)^{-1} \log_p \bar{v}(1 - \zeta_f^a),
\]
where \(\tau = \tau(\chi) = \sum_{a=1}^{f} \chi(a) \zeta_f^a\) is the Gauß sum associated to \(\chi\). Therefore the Solomon’s element \(\psi_F\) is obviously related to these values at 1.

These \(p\)-adic \(L\) functions are uniquely determined by this interpolation properties. Their original definition using Mahler development of partial zeta functions does not fully reveal their real nature. Later on Iwasawa discovered a far more concrete definition by using a sequence of Stickelberger elements to obtain a power series development of these functions. These power series are elements of \(\mathbb{Z}_p(\chi)[[T]]\).

**Definition 2.3.** — The Stickelberger element \(\xi_{Q, \zeta_f^{p^n}}\) related to \(\mathbb{Q}(\zeta_f^{p^n})\) is defined by the formula
\[
\xi_{Q, \zeta_f^{p^n}} = \frac{1}{fp^n} \sum_{a=1}^{f} \chi(a)^{-1} \left(\frac{f p^n}{2} - a\right) \sigma_a^{-1} \in \mathbb{Q}[\text{Gal} (\mathbb{Q}(\zeta_f^{p^n})/\mathbb{Q})].
\]

By definition the Stickelberger element related to \(K_n\) is the element deduced from \(\xi_{Q, \zeta_f^{p^n}}\) by restriction from \(\text{Gal} (\mathbb{Q}(\zeta_f^{p^n})/\mathbb{Q})\) to \(\Delta_{K_n}\).
\[
\xi_{K_n} = \frac{1}{fp^n} \sum_{a=1}^{f} \chi(a)^{-1} \left(\frac{f p^n}{2} - a\right) \sigma_a^{-1})_{|K_n} \in \mathbb{Q}[\Delta_{K_n}].
\]

One can then show that

\(^{(1)}\)completer la ref
1. Let \( \vartheta_{Q(\zeta f^n)} = (1 - (1 + fp)(\kappa^{-1}(1 + fp))^{-1}) \xi_{Q(\zeta f^n)} \). Then one has

\[
\vartheta_{Q(\zeta f^n)} \in \mathbb{Z}_p[\text{Gal}(Q(\zeta f^n)/\mathbb{Q})],
\]

so that its restriction \( \vartheta_{K_n} \) to \( \mathbb{Q}[\Delta_{K_n}] \) is also a \( p \)-adic integral element actually an element of \( \mathbb{Z}_p[\Delta_{K_n}] \).

2. The sequences \( (\xi_{K_n})_n \in \mathbb{N} \) and therefore \( (\vartheta_{K_n})_n \in \mathbb{N} \) are coherent with respect to restrictions maps along the towers \( Q(f^n)/Q(f) \) and \( K_\infty/K \).

As the \( \vartheta_{K_n} \) are coherent along \( K_\infty/K \) we may consider the limit \( \vartheta_{K_\infty} \in \Lambda(\Delta_{K_\infty}) \).

\[\textbf{Definition 2.4.} \quad \text{We may now define the three relevant Iwasawa’s power series attached to our Dirichlet character } \chi: \]

1. \( g(T, \chi) = S(\chi^*(\vartheta_{K_\infty})) \in \mathbb{Z}_p(\chi)[[T]] \);
2. \( h(T, \chi) = 1 - \frac{1 + pf}{1 + T} \in \mathbb{Z}_p[[T]] \);
3. \( f(T, \chi) = \frac{g(T, \chi)}{h(T, \chi)} \in \mathbb{Z}_p(\chi)[[T]] \).

Iwasawa in [Iwa69] discovered that these series would provide another definition of the functions \( L_p(s, \chi) \). This definition turned out to be far more enlightening than the original one.

\[\textbf{Theorem 2.5 ([Iwa69]).} \quad \text{Let } \chi \text{ be an even character of } \Delta_K. \text{ Then for all } s \in \mathbb{C}_p \text{ such that } |s| < \frac{pp^{-1/(p-1)}}{1} \text{ (and } s \neq 1 \text{ if } \chi \text{ is trivial) we have}
\]

\[
L_p(s, \chi) = f((1 + fp)^s - 1, \chi).
\]

\[\textbf{2.3. Algebraic side.} \quad \text{For all number field } F \text{ recall that}
\]

- \( X_F \) is the \( p \)-part of the class group of \( F \).
- \( \mathfrak{x}_k \) is the Galois group of the maximal abelian \( p \)-ramified \( p \)-extension of \( F \).

We \( X_\infty = \lim X_{F_n} \), \( \mathfrak{x}_\infty = \lim \mathfrak{x}_{F_n} \) (with respect to norms and restriction maps). These are finitely generated \( \Lambda(\Delta_F) \)-modules and are linked together by the exact sequence

\[
0 \to E_\infty \to U_\infty \to \mathfrak{x}_\infty \to X_\infty \to 0
\]

where

- \( E_\infty \) is the direct limit of the \( E_n = E_n \otimes \mathbb{Z}_p \) where the \( E_n \) are the global units of the \( F_n \).
- \( U_\infty \) is the direct limit of the semi-local units of the \( F_n \).
By the weak Leopoldt conjecture (here a theorem) the $\Lambda$-module $X_\infty$ is torsion. In particular, for all character $\chi$ of $\Delta_F$, the $\chi$-part $X_\infty^\chi$ is a finitely generated torsion $\mathbb{Z}_p(\chi)[[T]]$-module. We therefore may consider its characteristic series that we will denote $c.s.(X_\infty^\chi)$.

**Remark 2.6.** — It is a well known theorem of Iwasawa (see [Iwa59]) that $X_\infty^\chi$ has no non-trivial finite $\Lambda$-submodule. In particular it is annihilated by its characteristic series.

2.4. Main conjecture. — The main conjecture of Iwasawa relevant to this context may be stated as follows:

**Theorem 2.7 (Mazur-Wiles).** — For all character $\chi$ of $\Delta_F$ we have

$$s.c.(X_\infty^\chi)(T) = f^\sharp(T, \chi)$$

where, by definition, the Iwasawa’s involution $\sharp$ is given by

$$f^\sharp(T, \chi) = f\left(\frac{1 + fp}{1 + T} - 1, \chi\right).$$

The mirror Stickelberger series $f^\sharp(T, \chi)$ plays an important role here. Another (equivalent) statement of the main conjecture is the following equality of characteristic series:

$$s.c.(X_\psi^\infty)(T) = s.c.(U_\infty/C_\infty)^\psi(T),$$

where $C_\infty$ stands by definition for the direct limit of circular units of the $F_n$.

3. Solomon’s element

Recall that for an abelian totally real field $F$ such that $p$ is unramified in $F$ the Solomon’s element is defined by the formula

$$\psi_F := \frac{1}{p} \sum_{\delta \in \Delta_F} \log_p \left(\frac{1}{Q(\varphi_{\text{Gal}(Q_p(\zeta_p)/Q)}(1 - \zeta^\delta))} \delta^{-1}\right).$$

Our goal in that section is to use Coleman theory to relate this $\psi_F$ to the constant term of a power series that annihilates $X_\infty$.

3.1. Coleman’s theory. — This theory is essentially a local one and it stands for any unramified extension of $Q_p$. Let $O_v[[X]]$ be the ring of formal power series in one indeterminate $X$ with coefficient in $O_v$. It is important here not to mix the Coleman $X$ with the Iwasawa’s $T$. Let $O_v((X))$ be the field of fraction of $O_v[[X]]$. We will also denote $\Gamma^\times = \text{Gal}(Q_p(\zeta_p\infty)/Q_p)$; this notation is coherent with the previous notation $\Gamma = \text{Gal}(K_\infty/K)$.

The rings $O_v[[X]]$ and $O_v((X))$ are endowed with:

1. the action of a Frobenius $\varphi$ such that $f^\varphi(X) = f((1 + X)^p - 1).
2. a norm operator $N$ uniquely determined by $N(f^\varphi)(X) = \prod_{\xi \in \mu_p} f(\xi(1 + X) - 1).$
3. an action of $\mathbb{Z}_p[[\Gamma^\times]]$ which extends by linearity and continuity the action of $\Gamma^\times$ defined by $\gamma.(1 + X) = (1 + X)^{\kappa(\gamma)}$ for all $\gamma \in \Gamma^\times$ (recall that $\kappa$ is the cyclotomic character).

**Theorem 3.1 (Coleman).** — Let $(\alpha_n) = \alpha \in \lim_{n \to \infty} F_v(\zeta_p^n)^\times$ be a norm coherent sequence of local numbers. Then there exists a unique power series $f_\alpha \in \mathcal{O}_v((X))^\times$ such that

$$f_\alpha(\zeta_p^{n+1} - 1) = \alpha_n \text{ and } Nf_\alpha = f_\alpha^p.$$  

Let $Col : \lim_{n \to \infty} F_v(\zeta_p^n)^\times \to \mathcal{O}_v((X))$ be the map defined by

$$Col(\alpha) = (1 - \frac{\varphi}{p}) \log_p f_\alpha(X) = \frac{1}{p} \log_p \left(\frac{f_\alpha(X)}{f_\alpha^p(X)}\right).$$  

Then $Col$ induces an exact sequence

$$1 \to \mathbb{Z}_p(1) \to U_{1,\infty}^1 \xrightarrow{Col} \mathcal{R} \xrightarrow{\epsilon} \mathbb{Z}_p(1) \to 0.$$  

where

- $U_{1,\infty}^1$ is the direct limit of the principal units of $F_v(\zeta_p^n)$;
- $\mathcal{R}$ is a rank one free $\mathcal{O}_v[[\Gamma^\times]]$-module generated by $(1 + X)$;
- $\mathbb{Z}_p(1)$ is the direct limit of the $(\zeta_p^n)$;
- $\epsilon$ is the map $f \mapsto Df_X = 0$ where $D = (1 + X) \frac{d}{dX}$.

It is useful to consider another related exact sequence that is obtained by using the Mellin transform $\mathcal{L}$ of $Col$ with values in $\mathcal{O}_v[[\Gamma^\times]]$ instead of $\mathcal{R}$. Let us briefly restate this definition: We saw that $\mathcal{R}$ is a rank one free $\mathcal{O}_v[[\Gamma^\times]]$ generated by $(1 + X)$. Hence all element $g \in \mathcal{R}$ may uniquely be written as $g(X) = \hat{g}.(1 + X)$ with $\hat{g} \in \mathbb{Z}_p[[\Gamma^\times]]$. The element $\hat{g}$ (also denoted by $Mel(\hat{g})$) is called the Mellin transform of $g$ and we define $\mathcal{L} = Mel \circ Col$.

### 3.1.1. Formulas. —

We need to give a dictionary of formulas between $\mathcal{R}$ and $\mathcal{O}_v[[\Gamma^\times]]$. For all $\hat{g} = \sum a_n \gamma^n \in \mathcal{O}_v[[\Gamma^\times]]$ and all characters $\psi$ of $\Gamma^\times$ we note $\psi(\hat{g}) = \sum a_n \psi(\gamma)^n \in \mathbb{Z}_p(\psi)$. Recall that

$$\sum a_n \gamma^n \cdot (1 + X) = \sum a_n (1 + X)^{\kappa^n(\gamma)}$$  

where $\kappa$ is the cyclotomic character. It follows that for all $g \in \mathcal{R}, i \in \mathbb{Z}$

- $g(0) = 1(\hat{g})$ (where $1$ stands for the trivial character).
- $D^i g = Tw^i(\hat{g})$, where if $\hat{g} = \sum a_n \gamma^n$ then $Tw^i(\hat{g}) = \sum a_n (\kappa^i(\gamma) \gamma)^n$

One has in particular $(D^i g)_{X=0} = 1(Tw^i(\hat{g})) = \kappa^i(\hat{g})$. With all these formulas and notation the twin sequence of the exact sequence of Coleman is now

$$1 \to \mathbb{Z}_p(1) \to U_{1,\infty}^1 \xrightarrow{\mathcal{L}} \mathcal{O}_v[[\Gamma^\times]] \xrightarrow{\kappa} \mathbb{Z}_p(1) \to 0.$$
3.1.2. Induction. — Let \( L \) be the splitting field at \( p \) of \( F \), so that \( p \) is totally split in \( L/Q \) and that primes above \( p \) remains inert in \( F/L \). In order to recover global information for our field \( F \) from the local information collected in \( F_v \) one first need to induce along \( \text{Gal}(L/Q) \) and consider first the “semi-local” situation. We have \( \text{Gal}(F_v/Q_p) \cong \text{Gal}(F/L) \). Let us abbreviate \( \Delta_L = \text{Gal}(L/Q) \). We apply the (exact) functor \( - \otimes_{\mathbb{Z}_p[\text{Gal}(F/L)]} \mathbb{Z}_p[\Delta_F] \) to the sequence (1). We get (see also [Tsu99]) :

\[
0 \rightarrow \mathbb{Z}_p[\Delta_L](1) \rightarrow U_\infty \rightarrow \hat{O}_F[\Gamma^\times] \rightarrow \mathbb{Z}_p[\Delta_L](1) \rightarrow 0,
\]

where \( \hat{O}_F := O_F \otimes \mathbb{Z}_p \cong \mathbb{Z}_p[\Delta_F] \cong O_v \otimes_{\mathbb{Z}_p[\text{Gal}(F/L)]} \mathbb{Z}_p[\Delta_F] \cong O_v[\Delta_L] \) because \( O_v \) is Galois free (recall that \( p \) is unramified in \( F \)).

3.2. Cyclotomic units and Solomon’s element. — Consider the cyclotomic elements \( \eta_{F_n} := N_{Q(\zeta_{fp^n})/F_n}(1 - \zeta_{fp^n}) \). These numbers are cyclotomic units of \( F_n \) (recall that \( f \neq 1 \)) and they form a norm coherent sequence (this follows from the well known distribution relation satisfied by the numbers \( 1 - \zeta_n \)). This allows us to define :

**Definition 3.2.** — Soit

\[
\eta_{F,\infty} = \left( N_{Q(\zeta_{fp^n})/F_n}(1 - \zeta_{fp^n}) \right)_{n \in \mathbb{N}}.
\]

This is an element of \( \lim \leftarrow O_{F_n}^\times \) that may be seen as an element of \( U_\infty \) using the diagonal embedding.

We can now state and prove the key result of this note :

**Theorem 3.3.** — The Solomon’s element \( \psi_F \) is nothing more than

\[
\psi_F = \frac{1}{p-1} \mathds{1}(\mathcal{L}(\eta_{F,\infty})).
\]

**Proof.** — By the formulas given in 3.1.1 we have \( \mathds{1}(\mathcal{L}(\eta_F)) = \text{Col}(\eta_{F,\infty})(0) \), so that the theorem is equivalent to the identity

\[
(p-1)\psi_F = \text{Col}(\eta_{F,\infty})(0).
\]

We shall check this equality. First remark that the set of \( p \)-places of \( F \) is in one to one correspondence with \( \Delta_L \) via the map

\[
\Delta_L \leftrightarrow \{ v, \mid v \text{ is above } p \} \\
\delta \mapsto \delta^{-1}v.
\]

Hence for all \( \delta \in \Delta_L \), the \( \delta \)-component of \( \eta_{F,\infty} \) (seen as an element of \( U_\infty \)) is

\[
\iota_v(N_{Q(\zeta_{fp^n})/F_n}(1 - \zeta_{fp^n}))_n.
\]

Therefore we locally have (at the corresponding place \( \delta^{-1}v \)) that the associated Coleman series is

\[
\iota_v\left( \prod_{\sigma \in \text{Gal}(Q(\zeta_f)/F)} 1 - \zeta_f^{\delta^{-1}}(1 + X) \right).
\]
Therefore, and still at the place $\delta^{-1}v$, we obtain

$$\text{Col}(\eta_{f,\infty}) = \left(1 - \frac{\varphi}{p}\right)f_{\eta_{f,\infty}}(X)$$

$$= \frac{1}{p} \log_p \left(\prod (1 - \zeta_f^{\delta^{-1}}(1 + X))^{\frac{1}{p}}\right)$$

It follows that specializing at $X = 0$ we find that the $\delta$-component of $\text{Col}(\eta_{f,\infty})(0)$ is

$$\frac{p - 1}{p} \log_p v_{\mathbb{Q}(\zeta_f)/F}(1 - \zeta_f^{\delta^{-1}})$$

which is exactly $(p - 1)$ times the $\delta$-component of $\psi_F$. \(\square\)

We now prove that the power series related to $\eta_{f,\infty}$ annihilates $\mathfrak{X}_\infty$.

**Lemma 3.4.** — The element $L(\eta_{f,\infty})$ annihilates $\mathfrak{X}_\infty \otimes \mathcal{O}_v$.

**Proof.** — By the main conjecture, for all character $\chi$ of $\Delta_F$, the characteristic series of $\mathfrak{X}_\infty$ (which annihilates $\mathfrak{X}_\infty$) is the same as the one of $(U_\infty/C_\infty)^{\chi}$. But for all character $\chi$ of $\Delta_F$ the series $\chi(L(\eta_F))$ is a multiple of the characteristic series of $(U_\infty/C_\infty)^{\chi}$. This proves that $\chi(L(\eta_F))$ annihilates $\mathfrak{X}_\infty$ for all $\chi$. But the whole $\mathfrak{X}_\infty$ has no non-trivial $p$-torsion (Ferrero-Washington and Iwasawa for the finite $\Lambda$-module, [FW79, Iwa59]). The result follows. \(\square\)

Now our main result is

**Theorem 3.5.** — The Solomon elements $\psi_F$ annihilates the $\mathbb{Z}_p$-torsion $t\mathfrak{X}_F$ of $\mathfrak{X}_F$.

**Proof.** — Putting together the theorem 3.3 and the lemma 3.4 and Iwasawa co-invariant we see that $(p - 1)\psi_F$ annihilates $(\mathfrak{X}_\infty)_F$. But these Iwasawa co-invariant are known to be canonically isomorphic to $t\mathfrak{X}_F$. \(\square\)

**Remark 3.6.** — It is an interesting but technically difficult question to precisely relate the global integral Coleman power series $L(\eta_{F,\infty})$ with the global integral power series $\partial_{K_\infty} \in \Lambda(\Delta_K)$ and also to relate both to the Deligne-Ribet pseudo-measure. All these objects seems to be various avatar of the $p$-adic $L$ function but there is some discrepancy between them. We hope to be able to say more in a future work.

**References**


